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Growth, Lockdown and the Dynamics of the Covid-19 Pandemic[1]

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Abstract

We present an extended dynamic equilibrium framework describing the simultaneous evolution of aggregate economic variables, of the Covid-19 reproduction index and of the lockdown measure. We prove the existence and uniqueness of the stationary solution and characterize its stability features. We also perform some comparative statics exercises in order to test how epidemiological and economic variables are affected by public containment measures. Namely, more restrictive lockdowns accelerate the process of absorption of the pandemic but slows down economic activity. When public spending financed through income taxation and reinvested entirely in public spending on health is accounted, there is an optimal level of the tax rate that minimizes fatalities. We also characterize the optimal stationary lockdown trading-off health needs, which require reinforced containment measures, and economic needs, which instead require relatively high degrees of opening of the economic activity.

Keywords:

Growth; Lockdown; Public Health Policy; Reproduction Number

JEL Classification:

- **C62**: Mathematical and Quantitative Methods; Mathematical Methods; Programming Models; Mathematical and Simulation Modeling; Existence and Stability Conditions of Equilibrium
- **E21**: Consumption; Saving; Wealth
- **E23**: Production
- **I18**: Health: Government Policy; Regulation; Public Health

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1. Introduction

Starting from March 2020, every area of private and social life has been revolutionized: it was no longer possible to drink a coffee at the bar, take the train, go to the cinema or the theater, visit a friend or relative, take a walk and many more other things that until then were taken for granted. Within a few days we had to adapt to an "online" life based on smart working, distance learning and video calls to keep contacts alive.

This is because a new highly contagious virus completely unknown to the human immune system has appeared: SARS-CoV-2, better known as Covid-19, a virus that, like all its many variants, mainly affects the respiratory system, develops in different forms, from asymptomatic ones, to milder ones in which the symptoms are similar to those of a flu, up to the strongest forms in which it can also affect other organs and, in the most serious cases, lead to death.

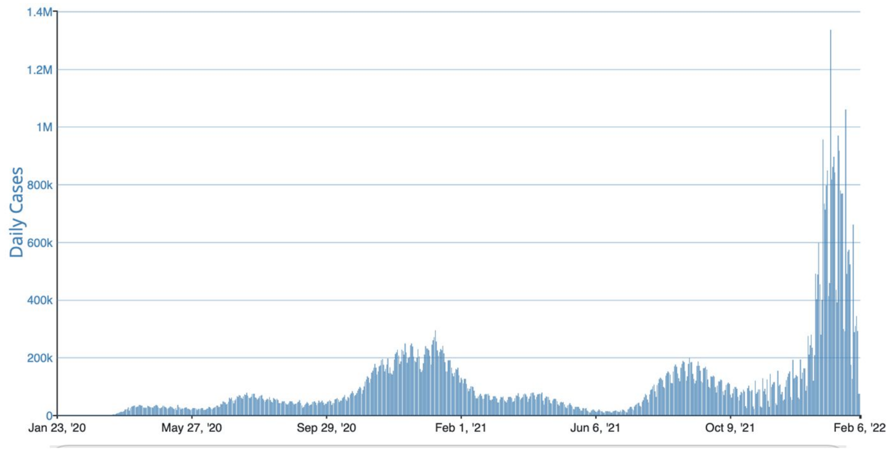
The first case of Covid-19 recorded in the world dates back to November 2019, in China, and then spread to every corner of the globe, so much so that, on March 11, 2020, the World Health Organization declared the State of Pandemic, forcing many governments to declare a state of emergency and to implement strict rules for the containment of the contagion, such as the limitation of individual movements, amassing and the suspension of many productive activities. In particular, almost all countries have introduced lockdown measures in order to limit human contacts.

Obviously, a lockdown, when it is restrictive enough, not only heavily affects private and social life, but also economic activity. For example, the size of the GDP and the employment rate are strongly affected by the implementation of the lockdown as well as by the degree of its severity. For this reason, at the time of launching restrictive health protocols, public authorities dramatically found themselves faced with a trade-off between health and economic needs. It should also not be forgotten that the health and economic aspects are even more closely linked than it could appear at first sight; for example, the size of the GDP dramatically affects the tax base and consequently the tax revenue used to finance the public health system with the inevitable repercussions on individual health.

In order to provide an overview of the impact of Covid-19 deasese on collective health and on aggregate economic activity, we focus on US data. In the graph below, as an example, it is reported the daily number of Covid-19 cases in US. The graph covers the period included between January 2020 and February 2022. As one can easily appreciate, there is a sequence of peaks appearing in the winter seasons, namely the months included between October and February. In such periods, the daily number of Covid-19 cases spans a whole interval included between 200.000 and, in the worst case, 1.3 million. In the hotter months, conversely, the Covid-19 cases undergo a contraction, confirming the hypothesis that the virus has a lower rate of reproduction when the outside temperature is higher.

It is also worthwhile to stress how the lockdown measures have been stricter in corresponding to periods of larger rates of contagion.

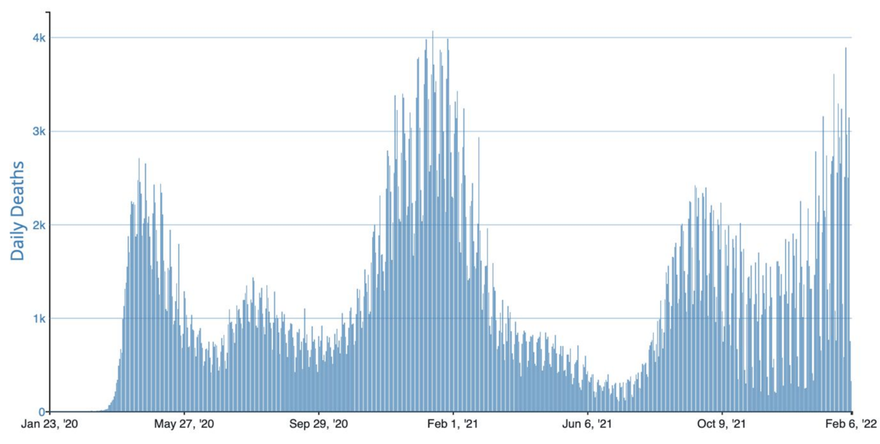
Figure 1: Daily Trends in Number of COVID-19 Cases in The United States Reported to CDC.



Source: CDC (Centre for Disease Control and Prevention).

As it is well known, for a given number of Covid-19 cases, there corresponds a given number of deaths. Such a number, in turns, depends upon different factors, e.g. on how aggressively the virus spreads around, on some environmental circumstances, on the efficiency of the health care system. In the following graph, it is reported the daily number of Covid-19 deaths in US. It is immediate to verify that its peaks overlap with the peaks of the graph reporting the daily number of Covid-19 cases. To such peaks, there corresponds a number of daily deaths included between 2.000 and 4.000.

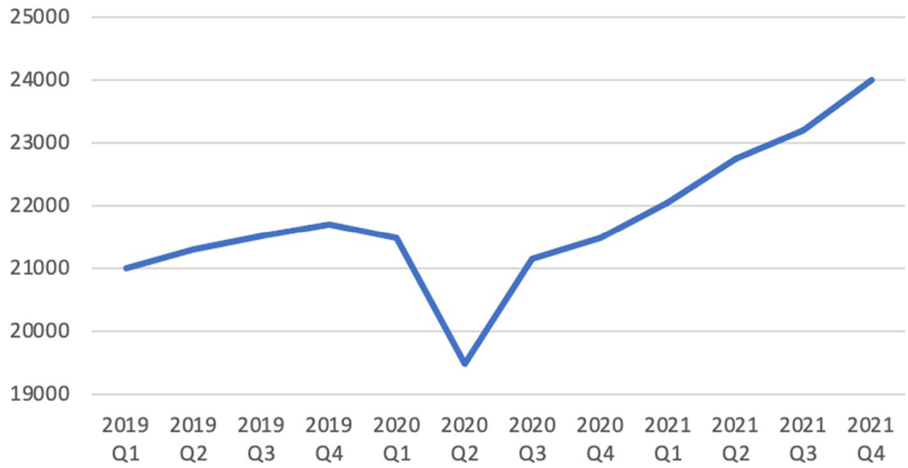
Figure 2: Daily Trends in Number of COVID-19 Deaths in The United States Reported to CDC.



Source: CDC (Centre for Disease Control and Prevention).

In the worst periods of the pandemic, in terms of infected and deaths, stricter sanitary protocols have been implemented in US. Among such protocols, there have been those promoting social distancing, as lockdown measures, even if the latter have been implemented at different degrees of severity. The immediate effects of the sanitary protocols have been the reduction of several economic activities incompatible with the requirement of social distancing and also the restriction on some usual consumption habits. As a consequence, the US GDP pattern has been strongly and negatively affected during the more dramatic pandemic phases. As it is depicted in the graph reported below, in correspondence to the first signals of pandemic, the GDP in US has first suffered a slowdown, then it started to sharply decline of even around a 10%. It then reached a minimum in the second trimester of 2020 and eventually underwent a slow reprise.

Figure 3: GDP US.



Source: Bureau of Economic Analysis.

Since the seminal work of Kermack and McKendrick (1927), mathematics has been widely applied/ to study the dynamics of epidemics and pandemics. An example is Gersovitz and Hammer (2004). On the other hand, one of the first attempts to couple SIR models with growth models is Goenka and Liu (2012; 2014). However, the recent Covid-19 pandemic crisis with its simultaneous repercussions on public health and economic activity has given rise to a renewed literature that skillfully combines epidemiology and economics in a dynamic perspective. For example, Eichenbaum et al. (2020) and Ng (2020), in order to identify the most effective anti-pandemic policies use a generalized SIR model in which agents react to confinement measures: in this way it is possible to determine, at an aggregate level, the transition matrices between one health status to another, such as from susceptible to infected and from infected to sick. Aum et al. (2021) find that lockdowns themselves may not present a clear trade-off between GDP and public health either. With regard to policies maximizing welfare, Alvarez et al. (2021) and Acemoglu et al. (2021) consider different lockdown policies and stress some technical difficulties arising when one must deal with non stationary sequences of lockdowns. Alvarez et al. (2021) appraise the optimal lockdown in terms of minimizing the fatalities of Covid-

19 and the related output costs of the lockdown. On the other hand, Acemoglou et al. (2021) study targeted lockdowns in a multigroup SIR model in order to provide a tractable quantitative analysis of optimal policy. Bosi et al. (2021) consider a dynamic framework with altruistic households and show that the optimal lockdown intensity increases in the degree of altruism.

To shed additional light to these still open and crucial questions, in this article we adapt the Ramsey (1928)-Cass (1965)-Koopmans (1965) growth model to simultaneously take into account the dynamics of capital, consumption, the disease reproduction index and the lockdown measure. After demonstrating the existence and uniqueness of the stationary solution, we perform comparative statics exercises in order to test how epidemiological and public policy parameters affect the stationary values of the endogenous variables. In this perspective, we find that a more restrictive lockdown measure allows a more efficient process of absorption of the pandemic but slows down economic activity. As regards the dynamic aspects of the model, we introduce a simple condition that ensures its stability, as well as characterize the qualitative typology of the latter. More in details, we find that the economy converges toward the steady state following damped oscillations.

The related mechanism is the following: when the contagion rate increases, a more restrictive lockdown is implemented and therefore production decreases. As a consequence, the pandemic slows down allowing in turn for a relaxation of the lockdown itself with positive effects on GDP. However, the contagion rate raises again and thus new restrictive measures become necessary, entailing an oscillatory, although deterministic, path.

Then we introduce public spending financed through income taxation and reinvested entirely in public spending on health, which has a decisive impact on the mortality rate of infected people. Next, we will calculate the optimal level of the tax rate that minimizes the number of deaths and subsequently the optimal lockdown that the Government should implement if it simultaneously takes into account the effects on GDP, and therefore on health expenditures, and on the reproduction index of the infection. Finally, the optimal lockdown parameter will be identified when the Government should be faced with a trade-off between health needs, which require reinforced containment measures, and economic needs, which instead require relatively high degrees of opening of the economic activity.

The remainder of the paper is structured as follows. In Section 2 we characterize the technology and the individual optimizing behavior while the subsequent Section 3 describes the dynamics of the reproduction number and of the lockdown measure. Section 4 provides the whole dynamic equilibrium and focuses on its stationary solution as well as on its stability features. Section 5 is devoted to the optimal taxation while in Sections 6 and 7 we characterize the optimal public policies trading off economic and health costs. Section 8 concludes the paper.

2. The Technology and the Individual Maximization Behavior

2.1 Technology

We will carry out our analysis from the point of view of a representative agent who in each period chooses how much to invest in capital and how much to consume out of his present income which is represented by the outcome of his production function employing his own stock of capital and his own labor effort to produce an homogeneous final good which can be in turn either consumed or invested. Notice that we are assuming that each individual behaves as an autonomous economic unit, i.e. he runs his own private "firm" inside which he works and employs his own capital stock.

In our model, however, in view of the pandemic emergency and the sanitary protocols implemented to tackle the former, we will assume that in a given period a variable lockdown index $\lambda \in [0,1]$ bounds agents to use only a fraction λ of their capital stock and of their labor effort for productive services. This in particular means that when λ is equal to zero, the whole amount of productive factors cannot be used for the scope of production; on the other hand, as soon as λ increases, more and more of the available inputs may be used for productive purposes; eventually, when λ =1, they can be completely exploited.

We consider a standard Cobb-Douglas Constant Returns to Scale technology employing capital and labor as productive inputs. If we assume a labor supply normalized to one, the per-period production function for the representative agent is then given by

(1) $\mathrm{y_t} = \frac{1}{\theta}(\lambda_t K_t)^{\theta}(\lambda_t L_t)^{1-\theta} = \frac{1}{\theta}\lambda_t k_t^{\theta}$

where $0 < \theta < 1$ is the share of capital in total income and K and L denotes, respectively, the capital endowment and the labor effort of the representative individual employed in the period under study. As a consequence, $k_{\mathrm{t}} = \frac{K_t}{L_t} = K_t$ denotes the intensive capital stock. A direct implication of such a framework is that the real wage ω and the real interest rate r are given, respectively, by

(2) $\omega = \lambda \frac{1}{\theta}(1-\theta)k^{\theta}$ and $r = \lambda k^{\theta-1}$

Finally, and without any loss in generality but for sake of simplicity in computations, we assume that capital fully depreciates from one period to another, i.e. $\delta = 1$, where δ is the depreciation rate. It is worth noticing that the removal of the hypothesis of capital full depreciation would not alter the results from a qualitative point of view.

2.2 – Individual Optimization Behavior

We assume that the economy is populated by a continuum defined on $[0,1]$ of identical individuals whose life-time horizon is infinite. However, due to the threat of the spread of the disease, in each period of their life they can become infected and, in the sequel, either become recovered or they can die.

Such a deviation from the standard Ramsey model should modify the individual behavior, e.g. by making the discount factor to depend upon the current spread of the disease, namely the current reproduction number. However, when modeling the individual behavior, we will make abstraction of the uncertainty agents are faced with, namely the probability of becoming infected and perhaps to die. Indeed, we will assume that each individual behaves as he were infinitely-lived; this hypothesis may be justified on the ground that people actually perceive the impact of the aggregate mortality rate on his life expectancy as being rather low and the lockdown measures as being quite efficient.

However, deaths are not ruled out and this implies that the size of the population is not constant, being the balance between births and deaths. In order to avoid complex aggregation procedures, we then assume that each individual owns and runs his private firm by using his own capital and by working in it. We assume that work is effectued outside and thus promotes the spread of the disease.

Since the labor supply is inelastic and normalized to one, under the technological and behavioral hypothesis above introduced, we can assume that the representative agent maximizes the discounted utilities of present and future consumption of the form:

(3) $\sum_{t=0}^{+\infty} \beta^t lnc_t$

where $0 < \beta < 1$ denotes the discount factor, subject to the sequences of budget constraints

(4) $c_t + k_{t+1} = \lambda_t \frac{1}{\theta} k_t^{\theta}, \;\; t = 0,1,2, \dots$

By maximizing (3) subject to (4), one immediately derives the Euler Equation, i.e. the optimal intertemporal smoothing behavior relative to periods t and $t+1$:

(5) $c_{t+1} = c_t\left[\beta \lambda_{t+1} k_{t+1}^{\theta-1}\right]$

Equations (4) and (5) fully describe the individual behavior in terms of sequences of consumption and investment, for a given lockdown policy represented by the sequence $\{\lambda_t\}_{t=0}^{+\infty}$. Notice that our unique departure from the standard Ramsey model is the presence in the production function of the lockdown measure which, as we are going to see, follows a specific dynamic by interacting with the pattern of the reproduction number of the disease. The usual transversality condition completes the characterization of the individual maximization problem:

(6) $\lim_{t \to +\infty} \beta^t k_t / c_t = 0$

3. Lockdown and Reproduction Number Dynamics

In this Section we introduce the epidemiological side of the model. More in details, we describe the evolution of the reproduction number and the dynamics of the lockdown measure implemented to tackle and curb the spread of the disease.

3.1 The reproduction number

The reproduction number R_t denotes the rate of contagiousness of a disease, i.e. states how many individuals on average are contaminated by a single infected within period$t$. In other words, if the reproduction number in period t is R_t and the size of the population infected is equal to I_t, the number of new infected in the following period will be $I_{t+1} = R_t I_t$. Obviously, the total number of infected is calculated net of the healed and the dead. It follows that when the reproduction number is larger than one, the infected will grow exponentially; on the other hand, if the reproduction number is kept lower than one, infected will exponentially decrease. It is worthwhile noting that, according to the OMS definition, a reproduction number larger than one gives rise to a situation of pandemic, meanwhile, when it is strictly lower than one, one refers to an epidemic.

In our work, we will assume that the reproduction number evolves as a positive function of its previous value, feature capturing a degree of persistence of the disease, and as a negative function of the lockdown measure (i.e. as a positive function of the degree of openness of the economic activity), since the latter is supposed to curb the spread of the disease. For sake of concreteness, we assume that the evolution of the reproduction number has the following law of motion:

(7) $R_{t+1} = aR_t + b\lambda_t + e$

with $a > 0$ and $b > 0$. As it is clear in (7), the reproduction number in period $t + 1$ depends positively both on the reproduction number of the previous period and on the degree of openness of economic activity, namely the number of worked hours, supposed to be effectued outside and thus promoting the transmission of the disease. This reflects, on the one hand, the fact that R_t follows an autonomous dynamics which is purely the fruit of epidemiological trends. On the other hand, it seems plausible to assume that the reproduction number increases as soon as the restrictive measures on economic activity implemented by the Government are softened (i.e. λ is made to increase): this in fact will entail more circulation of people, thus less social distancing and more people getting in close physical contact.

The reproduction number depends, in principle, also on an autonomous component e which embodies all the remaining variables with some incidence on the progression of the disease. However, for sake of simplification, we will assume e to be identically equal to zero. We thus obtain the following definitive dynamics for the reproduction number:

(8) $R_{t+1} = aR_t + b\lambda_t$.

The parameters a and b capture the importance on the spread of the disease attributed to, respectively, the past value of the reproduction number and the lockdown measure. An educated guess consists in assuming that there exists a natural absorption process for the reproduction number, i.e. that in correspondence of a complete lockdown, in the long run it converges to zero. Otherwise, it would be sufficient an arbitrarily small survenience of the disease to entail an unbounded pandemic even in presence of very restrictive sanitary protocols. On the ground of such considerations, we introduce the following mild assumption on the parameter a.

Assumption 1. a < 1.

As we will show in the sequel, the above restriction on a is very useful to ensure numerically significative features both for the steady state values of the model as well as for its stability. At the same time, notice that also the parameter b reflects a purely exogenous and "natural" feature, a purely epidemiological characteristic, although in practice it can be also the fruit of institutional, economic, behavioral and cultural factors. In any case, for the purpose of our analysis, b is taken as given both by individuals and Government. In the sequel, we describe how Government implements the sanitary protocols, namely according to which rule it fixes the lockdown measure.

3.2 The Lockdown Index

We define the lockdown index λ_t as the degree of openness of the economic system, namely one minus the lockdown amplitude. We assume that such an index is fixed by the Government on the ground of the past observed value of the reproduction number. For sake of concreteness, the evolution of the lockdown index is assumed to be described by the following equation:

(9) $\lambda_{t+1} = 1 - \gamma R_t$.

According to (9), the lockdown index in period t+1, λ_{t+1}, depends negatively upon the reproduction number R_t observed in period t. Notice that the parameter $\gamma > 0$ is chosen by the Government and reflects the degree of responsiveness of the lockdown index with respect to the previous period reproduction number. As a consequence, the Government will set γ on the ground of its decision of hardening or softening the lockdown measure.

From equation (9) one immediately verifies that when the reproduction number R_t is equal to zero, there will be no rationale for introducing any lockdown measure and thus λ_{t+1} will be identically equal to one. Of course, the lockdown index cannot be negative; this implies that it must always be $1-\gamma R_t > 0$ and therefore $\gamma < \frac{1}{R_t}$. As a consequence, in order to make consistent our model even in pandemic times ($R_t > 1$), the Government must fix γ strictly lower than one.

Assumption 2. $\gamma < 1$.

However, if the system converges toward plausible values for the reproduction number and the lockdown index, it may be possible to violate Assumption 1, since even if for some initial periods they falls out of their domain of definition, nevertheless their consistency is quickly reestablished as soon as enough time is allowed to pass.

4. The Dynamic System

The dynamics of the system is fully described by equations (4), (5), (8) and (9). It involves the evolution through time of k_t, c_t, R_t and λ_t. In order to be economically meaningful, such variables must remain non-negative all through the time. In period zero, from the individual perspective, the values k_0, R_0 and λ_0 are known, i.e. are predeterminated, while initial consumption c_0 must be chosen by the representative agent on the ground of her expectations about the future. This implies that the dynamic system will be locally determinate if and only if the dimension of the stable manifold of the steady state is equal or lower than three. Otherwise, would its dimension be equal to four, the steady state would be locally indeterminate, since there will be infinitely many choices for the initial consumption c_0 compatible with the convergence toward the steady state state and thus not violating the transversality condition. Our first task is now to prove the existence and uniqueness of the stationary solution of system defined by (4), (5), (8) and (9). Then, we will perform some comparative statics.

4.1 Steady State Analysis

A steady state of our economy is a non-negative vector (k, c, R, λ) solving equations (4), (5), (8) and (9) once one has got rid of the time index. Actually, such vector must solve the following system:

$$(10) \begin{cases} c + k = \lambda \frac{1}{\theta} k^{\theta} \\ 1 = \beta \lambda k^{\theta - 1} \\ R = aR + b\lambda \\ \lambda = 1 - \gamma R \end{cases}$$

Since the last two equations of (10) include only the variables R and λ, it is possible to solve them independently. Straightforward computations allow to obtain the following values, denoted with an asterisk, for the stationary reproduction number and the stationary lockdown index:

$$(11)\begin{cases} R^* = \frac{b}{1+\gamma b-a} \\ \lambda^* = \frac{1-a}{1+\gamma b-a} \end{cases}$$

It is immediate to verify that the stationary values R^* and $\lambda^*$of are unique and, under Assumption 1, strictly positive. Once one has derived R^* and λ^* one immediately obtains the unique stationary values for k and c, i.e. k^* and c^*:

$$(12)\begin{cases} k^* = \left(\beta \frac{1-a}{1+\gamma b-a}\right)^{\frac{1}{1-\theta}} \\ c^* = \frac{1}{\theta}\beta^{\frac{\theta}{1-\theta}}\left(\frac{1-a}{1+\gamma b-a}\right)^{\frac{1+\theta}{1-\theta}} \end{cases}$$

which, again under Assumption 1, are unique and strictly positive. All these results are gathered in the following Proposition.

Proposition 1. Under Assumption 1, dynamic system defined by (4), (5), (8) e (9) possesses a unique non-negative stationary solution $(k^*, c^*, R^*, \lambda^*)$.

From (11) and (12), one immediately verifies, that R^*, λ^*, k^* and c^* depend uniquely on the parameters θ, β, a, b and γ. It is worthwhile therefore performing comparative statics in order to appraise how the stationary values of the system do react when the structural parameters are made to vary.

4.2 Comparative Statics

First of all, it is straightforward to appraise that an increase in the share θ of capital in total income does not influence R^* and λ^* but uniquely increases the stationary capital stock k^* and the stationary consumption c^*. The intuition is immediate: since the labor supply is inelastically kept equal to one, an increase in the capital productivity does not affect the labor contribution to income but increases the capital one. As a result, the production possibility frontier shifts upward and it is possible to invest and consume more and more.

Let us now check the influence of the discount factor β on the stationary values of the system. In view of (11) and (12), one can see that it impacts only the stationary capital stock k^* and the stationary amount of consumption c^*. As a matter of fact, both k^* and c^* are increasing in β: such a result is quite standard since an increase in the degree of patience pushes agents to increase investment, to postpone consumption and thus to dispose in the long term of a larger stock of capital and of a larger amount of consumption.

The effect of an increase in the degree a of persistence of the disease in also quite intuitive. As one can easily check in (11), it increases the stationary reproduction number R^*. This in turn requires the implementation of a stronger lockdown measure, which yields to a lower stationary value for λ^*. In the limit case $a = 1$, it would be $\lambda^* = 0$ and $R^* = \frac{1}{\gamma}$. The associated impact on the stationary values k^* and $c^*$of the economic variables is therefore straightforward: a larger a will entail a lower capital and labor productivities and a slower capital accumulation process. Thus, as one may directly check in (12), there will emerge a lower stationary capital stock k^* as well as a lower sustainable stationary consumption c^*.

At the same time, since the parameter b captures the reactivity of the reproduction number R to the degree of openness λ of economic activity, its augmentation entails a larger stationary value R^*. This implies that the sanitary protocols must become more restrictive, entailing thus a lower stationary lockdown index λ^*. In the limit case $b = 0$, it will be $\lambda^* = 1$ and $R^* = 0$ since the lockdown measure now has no impact on the reproduction number R and thus there is no incentive for public authority to implement restrictive measures on economic activity; at the same time, the pandemic will tend to disappear according to its natural rate of absorption a. In the other other limit case $b \to +\infty$, the degree of openness of economic activity has an infinitely harmful effect on the evolution of the reproduction number. This implies that in order to curb the pandemic, it is necessary to set λ^* identically equal to zero in order to reach the minimum possible level for the reproduction number corresponding to $R^* = \frac{1}{\gamma}$. As a consequence, both capital and consumption will converge to zero.

Eventually, the effect of an increase of γ on the stationary values in (11) and (12) is quite intuitive. Indeed, since such parameters captures the reactivity of the lockdown measure to the reproduction number, its increase will make the former more effective. As a consequence, on the one hand the stationary reproduction number R^* will be lower; on the other one, the lockdown rule will be more effective yielding to a lower stationary value λ^* qnd thus of capital and consumption. In the limit case $\gamma = 0$, the stationary reproduction number would be $R^* = \frac{b}{1-a}$ and the lockdown index λ^* equal to one, since it has no effect in curbing the spread of the disease. In the opposite limit case $\gamma \to +\infty$, the infinitely strong reactivity of the lockdown measure to the spread of the disease will be able to drive R^* to zero by completely reducing economic activity, i.e. by setting $\lambda^* = 0$. Capital and consumption will thus be driven to zero.

4.3 Stability Analysis

In order to study the stability properties of the dynamic system described by equations (4), (5), (8) e (9) in terms of the evolution through time of k_t, c_t, R_t and λ_t, for given initial conditions k_0, R_0 and λ_0 for c_0 as jump variable, let us linearize it around its unique stationary solution given by (11) and (12). Straightforward although tedious computations yield the following linearized dynamics for the deviations from the steady state:

$$(13)\quad \begin{bmatrix} dk_{t+1} \\ dc_{t+1} \\ dR_{t+1} \\ d\lambda_{t+1} \end{bmatrix} = J \begin{bmatrix} dk_t \\ dc_t \\ dR_t \\ d\lambda_t \end{bmatrix}$$

where

$$(14)\ J = \begin{bmatrix} \frac{1}{\beta} & -1 & 0 & \frac{1}{\theta}(\lambda\beta)^{\frac{\theta}{1-\theta}} \\ A & B & C & D \\ 0 & 0 & a & b \\ 0 & 0 & -\gamma & 0 \end{bmatrix}$$

denotes the Jacobian matrix evaluated at the stead state and

$$A \equiv \left[\beta c\lambda(\theta-1)\left(\frac{1}{\lambda\beta}\right)^{\frac{2-\theta}{1-\theta}}\frac{1}{\beta}\right] < 0, B \equiv \left[1-\beta c\lambda(\theta-1)\left(\frac{1}{\lambda\beta}\right)^{\frac{2-\theta}{1-\theta}}\right] > 0, C \equiv -\left[\frac{c}{\lambda}\gamma\right] < 0,$$

$$D \equiv \left[\beta c\lambda(\theta-1)\left(\frac{1}{\lambda\beta}\right)^{\frac{2-\theta}{1-\theta}}\frac{1}{\theta}(\lambda\beta)^{\frac{\theta}{1-\theta}}\right] < 0.$$

The stability features of the steady state are fully appraised by the study of the norm and of the sign of the four eingeinvalues of J. Such eingenvalues are obtained as the four solutions μ of the Characteristical Polynomial

$$P(\mu) = det\begin{bmatrix} \frac{1}{\beta}-\mu & -1 & 0 & \frac{1}{\theta}(\lambda\beta)^{\frac{\theta}{1-\theta}} \\ A & B-\mu & C & D \\ 0 & 0 & a-\mu & b \\ 0 & 0 & -\gamma & 0-\mu \end{bmatrix} = 0.$$

By developing the Characteristical Polynomial we find the following expression:

$$P(\mu) = P_1(\mu)P_2(\mu) = 0$$

where

$P_1(\mu) = \mu^2 - \left(\frac{1}{\beta}+B\right)\mu + \frac{B}{\beta} + A$ and $P_2(\mu) = \mu^2 - \mu a + \gamma b$

The Characteristical Polynomial of order four is thus the product of two second order polynomials. This is the immediate consequence of the fact the the Jacobian J is semi-triangular: indeed, the last two lines describe an autonomous dynamics exclusively in terms of R_t and λ_t. It is thus possible to characterize separately the solutions of the two equations.

Let us first study the solutions of the first polynomial $P_1(\mu) = 0$. If we evaluate it at $\mu = 0$, after some simple manipulation, we obtain

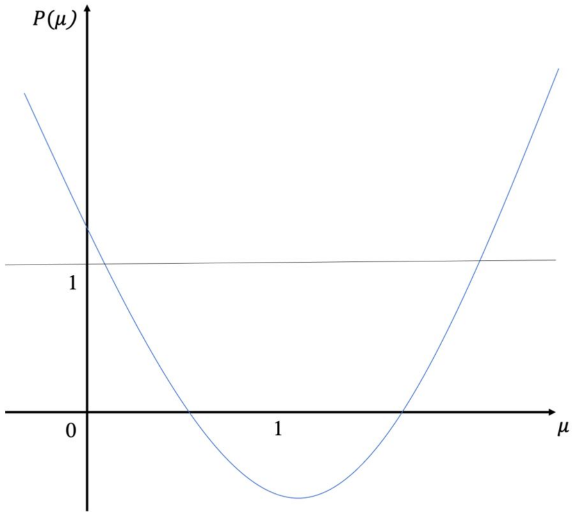
$$P_1(0) = \frac{B}{\beta} + A = \frac{1}{\beta} > 1.$$

At the same time, $P_1(\mu)$ evalaued at 1 can be written, after some straightforward computations, as

$$P_1(1) = \beta c\lambda(\theta - 1)\left(\frac{1}{\lambda\beta}\right)^{\frac{2-\theta}{1-\theta}} < 0.$$

These pieces of information, together with the fact that $\lim_{\mu\to-\infty} P_1(\mu) = \lim_{\mu\to+\infty} P_1(\mu) = +\infty$ and that $P_1(\mu)$ is continuous and defined on a connected set, are sufficient to ensure that there exists a real eingenvalue larger than one and a real eigenvalue included between zero and one, as it is shown in the following figure:

Figure 4: Local stability

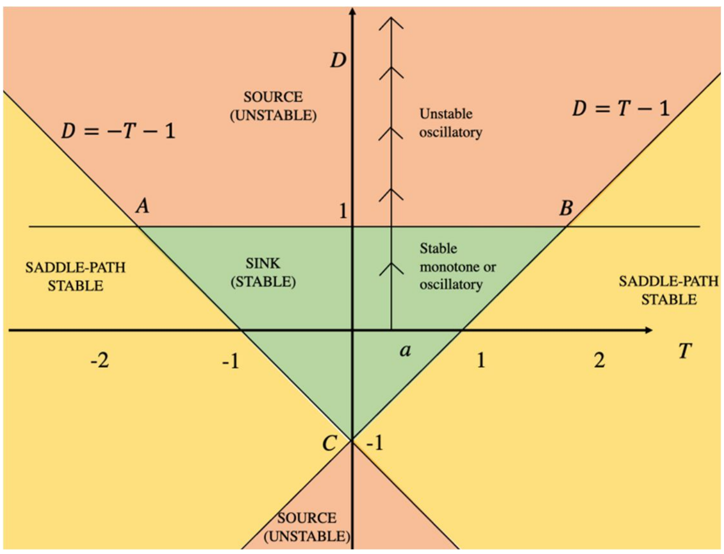


In order to study the two solutions of the other polynomial $P_2(\mu) = 0$, we adopt the geometrical approach presented in Grandmont et al. (1998) and Cazzavillan et al. (1998). Such a method consists in locating the Trace T and the Determinant D of the matrix

$$J_2 = \begin{bmatrix} a & b \\ -\gamma & 0 \end{bmatrix}$$

in the (T,D) plane. According to the location of the pair (T,D), there will arise different regimes for the stability of the steady state, as it is illustrated in the following Figure:

Figure 5: Local stability



The Trace T and Determinant D of J_2 are immediately derived as

$T = a$ and $D = b\gamma$

The Trace is thus equal to a and in view of Assumption 1, it is located in the abscissas' axis in the interval included between 0 and 1. In the ordinates' axis we can now plot the Determinant which is equal to $b\gamma$. If we fix a and b and let γ vary from zero to infinite, we obtain a vertical half-line whose origin is $(a, 0)$ and its end-point is $(a, +\infty)$. Such a half-line will cross the line $D = 1$ in correspondence to $\gamma = \frac{1}{b}$.

Therefore, for $\gamma < \frac{1}{b}$ there will be two stable eigenvalues (first real then complex conjugates) while, for $\gamma > \frac{1}{b}$, the two eigenvalues will become complex conjugates with norm larger than one, i.e. unstable. At $\gamma = \frac{1}{b}$ the norm of the two complex conjugate eigenvalues goes through one, and thus undergoes an Hopf bifurcation; as a consequence, as shown in Guckenheimer and Holmes (1983), an invariant closed curve will arise in a neighborhood of the stationary solution and R_t and λ_t will oscillate perpetually, although without ever going though the same points.

We can here draw some important considerations. Since, in order to curb the pandemic, the parameter γ must be set as larger as possible, but since in order to rule out instability it is required $\gamma \leq \frac{1}{b}$, the best value to tackle the pandemic, subject to the stability constraint, is $\gamma^* = \frac{1}{b}$ to which correspond the stationary values for R and λ:

$$\begin{cases} R^o = \dfrac{b}{2-a} \\ \lambda^o = \dfrac{1-a}{2-a} \end{cases}$$

Of course, the choice of γ, thorugh its impact on R and λ, has an impact also on k and c. Therefore, there could arise a trade-off between the need to tackle the pandemic (which requires a large γ) and the need to support economic growth (that, by constrast, requires a low γ). Such an analysis is actually the goal of the next Sections.

The stability analysis carried out can be summarized in the following Proposition that is immediately proved.

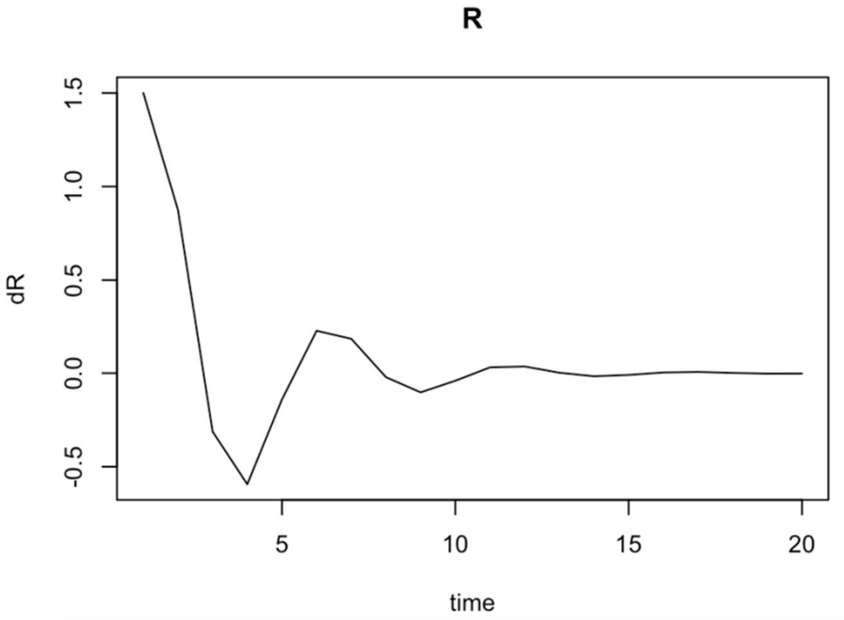
Proposition 2. Under Assumption 1, the unique steady state of system defined by (4), (5), (8) and (9) is

(i) Saddle path stable (locally determinate) for $\gamma < \frac{1}{b}$

(ii) Unstable (locally determinate) for $\gamma > \frac{1}{b}$

(iii) When for $\gamma = \frac{1}{b}$, a closed curve arises nearby (globally determinate).

In the following Figures we have plot the dynamics of the deviations of R and λ from their steady state values corresponding to the following standard calibration for the parameter values: β=0.99, $\theta = 0.33$, a=1/2, b=1/4, $\gamma = 2$. We have considered a length of time corresponding to 20. As it is possible to observe, both R and λ follow dampened oscillations and finally converges to their stationary solutions. In addition, they are perfectly countercyclical since when the contagion rate increases, one period later a more severe lockdown is implemented and vice versa. Of course, the path of capital and consumption is perfectly pro-cyclical with λ.



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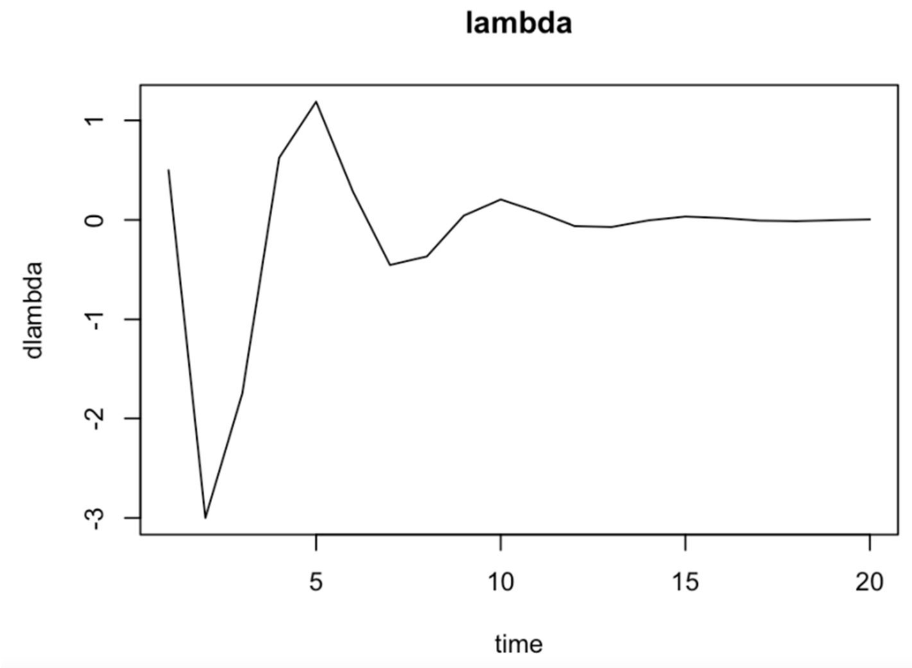


5. Public Spending in Health Services

In this section we introduce public spending. It is financed entirely through income taxes (there is therefore no dynamics of public debt) and is used by Government in health care in order to tackle the deadly cosequences of infected people. Such a goal is implemented, for example, by strengthening intensive care, increasing the purchase of medicines, more funds earmarked for research for treatment, etc. Since usually health care is rather a flow than a stock, we may assume that public spending fully depreciates from one period to another. If we introduce a flat tax $\tau \in [0,1\,]$ on total income $y,$ it follows that for the representative agent the disponible income in period t will now be

(15) $y_t \ = (1-\tau)\lambda_t \frac{1}{\theta} {k_t}^{\theta}$

while per-capita public spending will be

(16) $G_t = \tau\lambda_t \frac{1}{\theta} {k_t}^{\theta}$

By introducing the flat tax, the equation describing the dynamics of R and λ does not undergo any modification, as well as their steady state values remain those obtained in (11). Conversely, the introduction of the distortionary tax does reduce the stationary capital stock and consumption, which now are easily derived as

(17) $$\begin{cases} k^* = \left(\beta(1-\tau)\frac{1-a}{1+\gamma b-a}\right)^{\frac{1}{1-\theta}} \\ c^* = \frac{1}{\theta}(\beta(1-\tau))^{\frac{\theta}{1-\theta}}\left(\frac{1-a}{1+\gamma b-a}\right)^{\frac{1+\theta}{1-\theta}} \end{cases}$$

If the steady state values of k and c are affected by the introduction of the tax, the stability features of the system conversely does not undergo any modification, as the latter is now simply defined up to a less than one transformation of the discount factor.

We assume that the goal of the Government is to minimize the number of deaths. To this end, notice that in period t+1, the total amount of dead individuals M_{t+1} will be the sum of the previous deads, M_t, and the number of new deaths, D_t, that we assume to depend positively upon the number of infected people. Such a number, in turn, among the others factors, is dramatically related to the reproduction number R_t. On the other hand, the share of infected which die depends negatively upon spending in public health G_t. As a matter of fact, at the steady state the evolution of deaths can be represented as it follows:

(18) $M_{t+1} = M_t + D_t$

where

$$D_t = P\left((\psi R^*)^{\upsilon}\left(\tau\lambda^*\frac{1}{\theta}{k^*}^{\theta}\right)^{-\phi}\right)L_t.$$

L_t being the size of population in period t, thus of susceptibles, $0 \leq P(x) \leq 1, P(0) = 0, P'(x) > 0$, the probability for an individual to die, for a given fiscal policy, a given lockdown amplitude, a given income and a given reproduction number. Here $\psi > 0$ and $\upsilon > 0$ capture the measure of how the reproduction rate R of the disease may have fatal consequences and ϕ the capability of public spending in health care to curb the mortal consequences of the infection. By substituting in (18) and in the expression of D_t the stationary value k^*, one obtains:

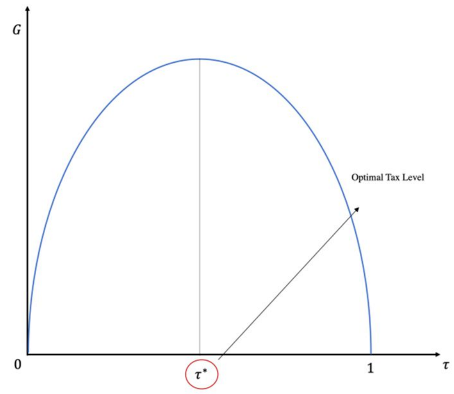
$$(19)\ M_{t+1} = M_t + P\left(\left(((\psi R^*)^{\upsilon})\tau\left(\frac{1}{\beta(1-\tau)\lambda\theta}\right)^{\frac{\theta}{\theta-1}}\right)^{-\phi}\right)L_t.$$

The problem for the Government is to find the optimal tax rate τ minimizing the number of new deaths, which in turn requires to minimize the probability for a given infected to die. Since the tax rate does not influence the stationary reproduction number R^* but only the amount of public health services, and the probability P is increasing in its argument, the Government must set τ in order to maximize the term

$$(20)\ \tau\left(\frac{1}{\beta(1-\tau)\lambda\theta}\right)^{\frac{\theta}{\theta-1}}$$

Equation (20), viewed as a function of τ, describes a bell-shaped curve starting from zero when $\tau = 0$, then assuming positive values in the interval $(0,1)$, and eventually becoming again equal to zero at $\tau = 1$. Therefore there will be a unique interior optimum for τ. This is a typical feature of the Laffer curve which trades off the rate effect and the tax base effect, as it depicted in Figure 6. Indeed, in the increasing part of the curve it is the rate affect to prevail, while, after having reached the maximum, it is the tax base effect to dominate, giving rise a downward sloping shape.

Figure 6: The Laffer Curve



By replacing in (20) the expression for λ^* and after some straightforward computations, one obtains the following expression for the optimal tax rate:

$\tau^* = 1 - \theta$

The optimal rate is equal to the share $1 - \theta$ of labor in total income, the same result found in Barro (1990) in which unbounded grow is driven by productive public spending under balanced public budget. Notice the larger α, the lower the production share destined for public expenditures.

As we have seen, another key parameter under the complete control of Government is γ, which fixes the degree of reactivity of the lockdown with respect to the evolution of the reproduction number. The optimal choice of γ is the objective of the next Section.

6. The Optimal Lockdown

We now study the optimal choice of the parameter γ. To this end, recall to mind that by fixing γ, the Government indirectly fixes also the stationary values of the system, defined in (11) and (12). Here again, we wonder which can be the optimal choice for γ if the Government aims at minimizing the number of deaths. This requires to minimize (20) with respect to γ. However, it is worth noting, that differently than in the previous case, now γ affects simultaneously R, λ and k.

If we substitute the stationary values that R^*, λ^*, k^* and c^*, minimazing the stationary increase of the number of deaths requires to minimize with respect to γ the following expression:

$$(21)\quad \left(\frac{b}{1+\gamma b-a}\right)^{\upsilon}\left(\left(\frac{1}{\left(1-\frac{\gamma b}{1+\gamma b-a}\right)(1-\theta)}\right)^{\frac{-\phi\theta}{\theta-1}}\right)$$

The problem can be rewritten as:

$$\min_{\gamma} b^{\upsilon}\left(\frac{1}{1+\gamma b-a}\right)^{\upsilon}(1-a)^{\frac{\phi\theta}{\theta-1}}\left(\frac{1}{1+\gamma b-a}\right)^{\frac{\phi\theta}{\theta-1}}$$

Since the term $(1-a)^{\frac{\phi\theta}{\theta-1}}$ does not include γ and by simplifying, one can solve

(22) $\min_{\gamma}\left(\frac{1}{1+\gamma b-a}\right)^{\upsilon+\frac{\mu\theta}{\theta-1}}$

It is immediate to verify that if $\upsilon+\frac{\mu\theta}{\theta-1}>0$, i.e. if $\upsilon(1-\theta)>\mu\theta$, then the solution of (22) requires the larger possible γ which, to be compatible with the stability of the state state, turns out to be $\gamma=1/b$. The optimal lockdown will then be $\lambda^{*}=\frac{1-\theta}{2-\theta}$ If, on the other hand, $\upsilon+\frac{\mu\theta}{\theta-1}<0$, i.e. if $\upsilon(1-\theta)<\mu\theta$, then the optimum is $\gamma=0$, to which corresponds the optimal lockdown $\lambda^{*}=1$. The intuition behind such results is the following. If the impact of the reproduction number on the mortality rate is larger than the corresponding disease contracting effect of public spending, then R must be kept as low as possible which in turn requires a large γ and thus the lockdown must be as severe as possible. On the contrary, were the impact of health care on the deaths containment effect to prevail with respect to the death enhancing effect of the reproduction number, the economic activity should be enhanced as much as possible; therefore λ should be maximized, i.e. γ set equal to zero.

7. Trade-off between GDP and Contagion Rate

The optimal lockdown measure found previously was calculated taking into consideration the sole objective of minimizing the number of deaths. However, there is another important objective to take into consideration: economic growth. It is in fact necessary that production be supported within the economy. In this way, a trade-off is created between economic objectives and health objectives: on the one hand it is necessary to loosen the lockdown to revive the economy, but on the other hand it is necessary to strengthen it to reduce infections. The problem therefore arises of finding the optimal lockdown level that efficiently mediates between economic and health needs.

To this end, let us assume that at the steady state the number of deaths depends solely upon the reproduction number and not upon public spending in health services, i.e. we set $\tau=0$. The number of deaths is then given by the following function:

$P((\psi R^{*})^{\upsilon})L_t$

It follows that the expression to be minimized becomes $(\psi R^{*})^{\upsilon}$. If one takes into account the need to support growth, by trading off the economic and health objectives, one must solve the following problem:

$$\max_{\gamma} -(\psi R^{\upsilon}) + \lambda \frac{1}{\theta} k^{\theta}$$

Notice that γ is chosen by the Government and its amplitude captures the degree of severity of the lockdown. By replacing the stationary values of the variables included in the above maximization problem, one has to solve

$$\max_{\gamma} -\left(\psi \left(\frac{b}{1+\gamma b - a}\right)^{\upsilon}\right) + \left(\frac{1-a}{1+\gamma b - a}\right)\left(\frac{1-a}{1+\gamma b - a}\right)^{\frac{\theta}{1-\theta}} (\beta\theta)^{\frac{\theta}{1-\theta}}$$

By computing the first order condition and by rearranging it, one can write

$$(1+\gamma b - a)^{-\upsilon+1+\frac{\theta}{1-\theta}} = b^{-2}\psi\upsilon b^{1-\upsilon}(\beta\theta)^{\frac{\theta}{1-\theta}}\left(\frac{\theta}{1-\theta}+1\right)(1-a)^{\frac{\theta}{1-\theta}}\big(b(1-a)\big)$$

By solving the above expression for γ one obtains:

$$\gamma^{*} = \frac{a-1+\sqrt[-\upsilon+1+\frac{\theta}{1-\theta}]{b^{-\upsilon}\psi\upsilon(\beta\theta(1-a))^{\frac{\theta}{1-\theta}}}}{b}$$

Notice that as soon as γ^{*} becomes larger, one has that it is its impact on the contagion rate to prevail agoast its impact on growth. In addition, if $\gamma^{*} < 0$, the obtimal γ would be equal to zero and the optimal lockdown equal to $\lambda^{*} = 1$. Such a case corresponds indeed to a stronger impact of the lockdown on economic growth rather than on the contagion rate. In addition, let us recall to mind that the stability condition requires γ lower than $1/_{b}$. Se γ^{*} is larger than such a value, Government would set γ exactly equal to $1/_{b}$ to which it corresponds a lockdown $\lambda^{*} = 1 - \frac{1}{b}R$. Eventually for solutions included between 0 and $1/_{b}$, it will corresponds a viable optimal strategy in terms of lockdown policy trading-off the sanitary and the economic objectives.

8. Conclusions

Following the appearance, in the first months of 2020, of the SARS-CoV-2 virus, better known as Covid-19, governments around the world have implemented drastic restrictive measures to contain its spread. One of the strongest measures that have been adopted is the lockdown, a limitation on the production and circulation of goods and subjects.

In this article, a dynamic equilibrium model has been presented in which the aggregate economic variables, the infection reproduction index and the lockdown index are simultaneously taken into account, in order to adapt the economic context with the limitation to the performance of the economic activity made necessary by the pandemic crisis.

We have established the existence and uniqueness of the stationary solution and we have introduced a simple condition that ensures its stability. We then characterized the optimal income taxation that maximizes the tax revenue to be allocated to healthcare spending. Next, we focused on the most effective lockdown measures in order to minimize economic losses and costs in terms of collective health.

Contrary to many similar studies, we have considered non-stationary lockdown sequences, as we postulate a reactivity of the containment measures with respect to pandemic conditions. However, in identifying the optimal lockdown, we limit ourselves to its long-term value. The identification of the entire dynamic trajectories, on the other hand, would require further analytical efforts which are difficult to treat due to the presence of non-convexities, as noted by Alvarez et al. (2021) and Acemoglu et al. (2021). This analysis could be a good insight for future research, together with the formalization of discount rates influenced by life expectancy and the introduction of non-linear lockdown feedback rules.

References

Alvarez, F., D. Argente, F. Lippi (2021). A Simple Planning Problem for COVID-19 Lock-down, Testing, and Tracing. *American Economic Review: Insights*, 3 (3): 367-82.

Acemoglu, D., V. Chernozhukov, I. Werning, and M. D. Whinston. (2021). Optimal Targeted Lockdowns in a Multigroup SIR Model. *American Economic Review: Insights*, 3 (4): 487-502.

Argente F., Lippi D. (2020), *A simple planning problem for COVID-19 lockdown*. National Bureau of Economic Research.

Barro, R. (1990). Government spending in a simple model of endogenous growth. *Journal of Political Economy* 98, 103–125.

Bosi, S., Camacho, C., D. Desmarchelier (2021). Optimal lockdown in altruistic economies. *Journal of Mathematical Economics* 93.

Cass D. (1965). Optimum growth in an aggregative model of capital accumulation. *Review of Economic Stududies* 32 , 233-240.

Cazzavillan, G., Lloyd-Braga, T., P. Pintus (1998). Multiple steady states and endogenous fuctuations with increasing returns to scale in production. *Journal of Economic Theory* 80, 60-107.

Eichenbaum, M. S., Rebelo, S., M. Trabandt (2020). The macroeconomics of epidemics. National Bureau of Economic Research No. w26882.

Gersovitz M., Hammer J.S. (2004). The economical control of infectious diseases. *Economic Journal* 114, 1-27.

Grandmont, J.M., (1985). On endogenous competitive business cycles. *Econometrica* 53, 995-1045.

Grandmont, J.M., (2008). Nonlinear difference equations, bifurcations and chaos: An introduction. *Research in Economics* 62, 122-177.

Grandmont, J.M., Pintus, P., de Vilder, R. (1998). Capital-labor substitution and competitive nonlinear endogenous business cycles. *Journal of Economic Theory* 80, 14-59.

Goenka A., Liu L. (2012). Infectious diseases and endogenous fluctuations. *Economic Theory* 50, 125-149.

Goenka A., Liu L. (2014). Infectious diseases and economic growth. *Journal of Mathematical Economics* 50, 34-53.

Gonzalez-Eiras M., Niepelt D. (2020). On the Optimal "Lockdown" during an Epidemic. Center for Economic Studies.

Guckenheimer, J., Holmes, P. (1983). Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer, New York.

Guerrieri V., Lorenzoni G., Straub L., Werning I. (2020), *Macroeconomic implications of covid-19: Can negative supply shocks cause demand shortages?* Technical report. National Bureau of Economic Research.

Kermack W. O., McKendrick A. G. (1927). A contribution to the mathematical theory of epidemics. Royal Society of London.

Koopmans T. (1965). On the concept of optimal economic growth The Econometric Approach To Development Planning, Rand-McNally, Chicago, 225-287.

Lastrapes W., VanHoose D., Wang P. (2020), To lockdown? When to peak? Will there be an end? A macroeconomic analysis on COVID-19 epidemic in the United States. *Journal of Macroeconomics* 65.

Ng, W. L. (2020). To Lockdown? When to Peak? Will There Be an End? A Macroeconomic Analysis on COVID-19 Epidemic in the United States. *Journal of Macroeconomics*, 65, 103-230.

Ramsey F.P. (1928). A mathematical theory of saving. *Economic Journal* 38, pp. 543-559.

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