Conditional Risk-Based Portfolio*

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Abstract

The risk-based investment strategies, such as Minimum Variance, Maximum Diversification, Equal Risk Contribution, Risk Parity, etc. share the common feature of being based on a risk measure of the asset returns, typically the covariance matrix. When one comes to implement these strategies, the standard approach consists in using an unconditional covariance matrix, simply estimated by the sample covariance matrix of past returns over a rolling window. An alternative consists in using a conditional covariance matrix that depends on information available to date. In this paper, we propose the first unifying and systematic comparison framework for the unconditional and conditional risk-based investment strategies. We compare their out-of-sample performances in terms of risk, returns and turnover (trading volume) with 4 criteria across 4 empirical datasets. Our results show that conditional risk-based strategies do not improve the out-of-sample Sharpe ratios, logically increase the turnover, but they reduce the ex-post risk.

Keywords: Risk-based investment strategies, Conditional risk measures

JEL classification: G10, G11.

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1 Introduction

Risk-based portfolio strategies, such as Minimum Variance (MV), Maximum Diversification (MD), Equal Risk Contribution (ERC), Risk Parity (RP), etc., are largely used by the asset management industry within many popular investment vehicles (smart beta ETF, mutual fund, etc.). Jurczenko et al. (2013) identify some of these uses in multi-asset allocation for more robustness in strategic decisions; in equities as alternatives to market capitalization benchmarks, which are heavily concentrated in a few stocks and significantly biased towards overvalued stocks and sectors; in the smart beta exchange-traded funds (ETFs) industry, and so on. Considering only RP strategies, the asset under management (AUM) of related investments is estimated at $400Bn (Financial Times, 2015), and in a recent survey of quantitative investment managers (800 clients of JPM in US and Europe), Kolanovic et al. (2015) found that 50% prefer a Risk Parity approach, versus 15% for traditional fixed weights, 20% Markowitz mean-variance optimization, and 20% active asset timing.[1]

Such a success is largely explained by the main common feature of these strategies. Whatever their definition and objective, the risk-based strategies do not require to forecast the expected returns and only rely on the estimation of a risk measure (volatility, Value-at-Risk, etc.) for the portfolio returns. Indeed, it is well-known that the traditional mean-variance optimization turns out to be an “estimation-error maximization" (Michaud, 1989). In particular, it relies on the estimation of expected returns which are notoriously unstable and hard to predict (Merton, 1980). Consequently, the mean-variance portfolios are extremely sensitive to the estimation errors in means (Frankfurter et al., 1971; Chopra and Ziemba, 1993; Kan and Zhou, 2007). In contrast, risk-based strategies only require the estimation of risk and dependencies between

[1]Kolanovic et al. (2015) mention a total AUM for the RP strategies equal to $500Bn (40% of these assets are allocated to equities).

assets, generally through the covariance matrix of asset returns. This is the case for the MV portfolio, i.e. the portfolio with smallest feasible variance which can be constructed from the available securities. Similarly, the covariance matrix is the only required input to derive the MD allocation which maximizes the ratio between undiversified and diversified volatility of the portfolio (Choueifaty and Coignard, 2008). It is also the case for the ERC strategy (Maillard et al., 2010) which defines a different approach to diversification, by spreading the ex-ante total risk equally among the portfolio components.[2]

Hence, all risk-based strategies require an estimate of the covariance matrix Σ, at each rebalancing date. In practice, asset managers generally consider the *unconditional* covariance matrix, and an estimate simply defined by the sample covariance matrix of past returns over a rolling window. However, an alternative consists in using a *conditional* covariance matrix that depends on information available to date.

In this paper, we propose the first systematic comparison of *unconditional* and *conditional* risk-based investment strategies. Our goal is to evaluate the out-of-sample relative performance of these strategies, in terms of ex-post returns, ex-post risk, and portfolio turnover. Both unconditional and conditional approaches have pros and cons. The *unconditional* approach has the great advantage of being model free and simple to implement, since no structure is imposed on the covariance matrix. However, it suffers from various drawbacks which have been largely documented in the literature devoted to mean-variance optimization. First, it implies to assume that the asset returns are stationary and Σ is constant over time. Second, the sample covariance matrix is a reliable estimator as long as the sample size is much greater than the number of assets. If this condition is not fulfilled, robustification methods are necessary (see

[2]The RP strategy (Qian, 2005; Roncalli, 2013) can be viewed as a generalization of the ERC in which the risk contributions are set equal to a risk budgeting target, not necessarily equal to $1/n$.

for instance Ledoit and Wolf (2004) and Ledoit and Wolf (2012), among others). Finally, as noticed by Martellini et al. (2014), the estimate depends on a particular historical scenario, and may not be fully representative of the true distribution of returns. In particular, the size of the estimation window plays a crucial role: increasing the window size diminishes the weight of recent information, and makes the covariance estimates less reactive to new information. At the limit, they are time-invariant. On the contrary, a small estimation window may induce estimation problems, as the covariance estimates are less robust, especially when the number of assets is large. Conversely, *conditional* approach implies to specify a model, typically a multivariate GARCH-type model, for the dynamics of the conditional covariance matrix (see Bauwens et al. (2006) for a survey). Consequently, the variance forecasts, and ultimately the optimal asset allocations, are likely to be affected by potential model misspecification and estimation errors.[3] However, conditional risk-based strategies have some advantages. The portfolios are likely to be more reactive as covariance estimates immediately incorporate new information. Thus, they should imply less ex-post risk than corresponding unconditional portfolios, even if the turnover (trading volume) is likely to be higher. The potential gains in terms of return performances of the conditional approach are a priori unclear.

Here, we propose a systematic and unifying comparison framework for the 3 most popular risk-based investment strategies, namely MV, ERC, and MD. For each of them, we compare the out-of-sample performance of the unconditional and conditional risk-based optimal portfolios, using the following 4 performance criteria: (1) the out-of-sample Sharpe ratio, (2) the ex-post return-losses, (3) the turnover, and (4) the ex-post portfolio volatility. We also consider the equally weighted portfolio as a benchmark, as in DeMiguel et al. (2009). The conditional risk-

[3]Ardia et al. (2017) recently propose an evaluation, based on Monte Carlo simulations, of the impact of covariance misspecification in risk-based portfolios.

based allocations are computed from the out-of-sample covariance matrix forecasts obtained from a DCC-GARCH model (Engle, 2002), since this model is widely used and is considered nowdays as a benchmark among the MGARCH models. Applying a rolling-window estimation procedure, both for the DCC parameters and the sample covariance matrix of past returns, we consider estimation windows of different lengths (500 or $1,000$ observations, respectively) and different forecasts horizons. The forecast horizons are determined by the portfolio rebalancing frequencies, which are fixed at 1 day, 1 week or 1 month. We apply our comparison procedure on 4 datasets which have been used by DeMiguel et al. (2009) in their seminal paper, but with extended data to the end of 2016. In order to conduct our study with the most favorable sampling frequency for conditional heteroskedasticity (ARCH effects), and ultimately for the conditional approach, we use daily returns for our estimation procedure.

Even within such a favorable framework, our empirical results are mitigated for the conditional approach. First, conditional risk-based strategies do not improve the out-of-sample Sharpe ratios obtained with basic unconditional approaches. We only observe that increasing the estimation windows' length tends to deteriorate the performance of unconditional portfolios, and as consequence to improve the relative performances of conditional approaches. The rebalancing frequency has no clear-cut influence on our diagnostic. Second, as expected, the turnovers of the conditional strategies are higher than those of the corresponding unconditional strategies. Third, the only gain obtained with conditional approaches is a decrease of ex-post risk. Whatever the dataset considered, the window length, and the rebalancing frequency, the out-of-sample variance of the portfolio returns always decreases with conditional approaches.

Our paper contributes to the literature devoted to risk-based investments, and especially to the literature dealing with the issue of their historical performance (Chow et al. (2011), Leote de

Carvalho et al. (2012), Choueifaty et al. (2013), among others). These studies are generally based on a comparison of the risk-based portfolio returns, with the returns of various alternative portfolios, including market-capitalization portfolios. However, these historical assessments only rely on unconditional approaches. Our results show that the conclusions of this literature should not be affected when considering conditional approaches. To the best of our knowledge, Martellini et al. (2014) is the only study that proposes a conditional strategy for risk-based investment. The authors introduce three distinct conditional RP strategies, explicitly designed to optimally respond to changes in state variables that have been used in the literature as proxies for the stochastically time-varying opportunity set. These strategies are based on three downside risk measures, namely the semi-variance, Value-at-Risk, and expected shortfall. They conclude to the superiority in various economic regimes of such conditional RP strategies with respect to standard unconditional RP techniques, based on unconditional volatility. Their goal and methodological approach is different from ours. First, they only consider the RP strategy and not the other risk-based investment strategies. Second, they propose new RP conditional strategies, based on downside risk measures. Our goal is different. It consists in assessing, for the same strategy, the gains associated to the use of a conditional covariance matrix instead of the traditional covariance matrix. Hence, our results have also direct implications for the asset management industry. They confirm the merits of the standard approach based on unconditional risk measures, currently used by practitioners, and its performances in terms of returns and turnover. But, they highlight the potential gains of using a conditional approach in terms of ex-post risk.

The rest of the paper is organized as follows. In Section 2, we introduce the notations and present the risk-based strategies. In Section 3, we detail the implementation of the unconditional and conditional approaches, and the estimation methodology. We also detail our unifying comparison framework and the criteria used to assess the out-of-sample performance of the portfolios. In Section 4, we describe the datasets. In Section 5, we conduct our empirical analysis and display our main takeaways. We summarize and conclude our paper in Section 6.

2 Risk based strategies

In this section, we present the main risk-based investment strategies and introduce the distinction between conditional and unconditional risk-based allocations. Consider a universe of n assets and denote by $r_t = (r_{1t}, ..., r_{nt})$ the n-dimensional vector of returns at time t. We denote $\omega_t = (\omega_{1t}, ..., \omega_{nt})'$ the n-dimensional vector of portfolio weights. For sake of simplicity, we impose no short-selling for all the strategies, meaning that $\omega_t \geq 0$. As usual, we assume that the sum of the weights is equal to 1, i.e. $e'\omega_t = 1$ with e the unit vector. Denote by σ_i^2 the variance of asset's return i and by σ_{ij} the covariance between asset i and j for $i \neq j$. Finally, let Σ be the covariance matrix of r_t and $\sigma_p = (\omega'\Sigma\omega)^{1/2}$ the standard deviation (volatility) of the portfolio.

Following Jurczenko et al. (2013), we define a risk-based portfolio as the allocation ω^* satisfying the following optimization program:

$$\omega^* = \arg\min_{\omega} D(f(\omega_i; \Sigma, \gamma, \delta)) \\ \text{u.c.} \begin{cases} \sum_{i=1}^{n} \omega_i = 1 \\ \omega_i \geq 0 \end{cases} \tag{1}$$

where $f(\omega_i; \Sigma, \gamma, \delta) = \frac{\omega_i^\gamma}{\sigma_i^\delta} \times \frac{\partial \sigma_p}{\partial \omega_i}$, with $\gamma \geq 0$ and $\delta \geq 0$ two parameters, $D(.)$ a dispersion metric such as the standard-deviation or the mean absolute deviation. The term $\partial\sigma_p/\partial\omega_i$ represents the marginal risk contribution of the i^{th} asset, which corresponds to the sensitivity of the portfolio volatility to a small change in the weight of asset i. So, the function $f(\omega_i; \Sigma, \gamma, \delta)$ can be interpreted as a "modified risk contribution" of asset i. Independently from the dispersion measure $D(.)$, solving program (1) gives the folling first order conditions:

$$\frac{\omega_i^{\gamma}}{\sigma_i^{\delta}} \times \frac{\partial \sigma_p}{\partial \omega_i} = \frac{\omega_j^{\gamma}}{\sigma_j^{\delta}} \times \frac{\partial \sigma_p}{\partial \omega_j} = \tau \qquad \forall (i,j) = (1, 2, ..., n), \tag{2}$$

with $\sum_{k=1}^{n} \omega_k = 1$ and $\omega_k \geq 0 \; \forall i = (1, 2, ..., n)$.

This optimization program encompasses the well-known risk-based strategies such as Minimum Variance (MV), Equal Risk Contribution (ERC) and Maximum Diversification (MD) given the values of parameters δ and γ (see Appendix A for more details). The existence and unicity of the optimal portfolios are discussed in Appendix B.

One advantage of risk-based strategies is that they only depend on portfolio risk, as measured by its volatility and ultimately by the covariance matrix Σ. In industry and academic literature, these strategies are generally based on the unconditional covariance matrix $\Sigma = \Sigma_u$. Whatever the risk-based strategy considered, the optimal allocation ω_u^* is then defined as a function $\lambda(.)$ with[4]

$$\omega_u^* \equiv \lambda\left(\Sigma_u; \delta, \gamma\right). \tag{3}$$

An alternative consists in using the conditional covariance matrix $\Sigma = \Sigma_{c,t} \equiv \mathbb{V}(r_t | \mathcal{F}_{t-h})$ where $\mathcal{F}_{t-h}$ is the information set available at time $t-h$, for $h \geq 1$. The conditional approach allows to take into account the changes in economic environment and is generally more responsive to economic and financial news. In this case, the optimal allocation $\omega_{c,t}^*$ is defined by the same

[4] For most of cases, the functional $\lambda(.)$ does not have a closed-form expression. The optimal portfolio is then the solution of a numerical optimization.

functional $\lambda(.)$, but is time-varying, with

$$\omega_{c,t}^{*} \equiv \lambda\left(\Sigma_{c,t};\delta,\gamma\right). \tag{4}$$

The goal of our paper is to assess the advantages of the two approaches and to compare the relative out-of-sample performances of the portfolios ω_u^* and $\omega_{c,t}^*$. To do so, we focus on 3 risk-based strategies, namely MV, ERC and MD. We now present these strategies in details. For ease of presentation, in the sequel we do not make the distinction between conditional and unconditional approaches, and we denote by Σ the covariance matrix.

2.1 Minimum variance

Under the MV strategy, we choose the allocation that minimizes the variance of portfolio return. This widely used strategy is the most popular among all the risk-based strategies. The MV portfolio, denoted ω^{MV}, is the result of the following optimization program:

$$\begin{aligned} \omega^{MV} = &\ \arg\min_{\omega}\ \omega'\Sigma\omega \\ &\text{u.c.} \begin{cases} e'\omega = 1 \\ \omega \geq 0 \end{cases}. \end{aligned} \tag{5}$$

It is straightforward to show that the first order conditions (FOC) of the MV program (5) and the general risk-based allocation program (Equation 1) are equivalent for $\gamma = 0$ and $\delta = 0$. In our case, there is no closed-form solution for ω^{MV} and the optimal allocation has to be determined numerically. However, when no constraint is imposed on the short-selling, the optimal allocation ω^{MV} has a closed-form expression given by

$$\omega^{MV} = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e}. \tag{6}$$

Since the MV portfolio minimizes the risk (measured by the volatility), it may induce a high concentration in the less risky assets.

2.2 Equal risk contribution

Contrary to the MV strategy, the ERC is risk diversification strategy defined in terms of risk contributions. The first formal analysis of the ERC portfolio was given by Maillard et al. (2010), who establish its existence and uniqueness, derive a number of analytical properties and propose numerical algorithms to compute the portfolio. The idea of ERC is to find a risk-balanced portfolio such that the risk contribution is the same for all assets of the portfolio. The risk contribution of asset i, denoted by $S_i(\omega, \Sigma)$, is defined as the share of total portfolio volatility attributable to that asset. Formally, we have

$$S_i(\omega, \Sigma) = \frac{\omega_i}{\sigma_p} \times \frac{\partial \sigma_p}{\partial \omega_i} = \omega_i \frac{(\Sigma\omega)_i}{\omega'\Sigma\omega} \tag{7}$$

where $(\Sigma\omega)_i$ is the i^{th} row of the n-dimensional vector $(\Sigma\omega)$. The risk contribution is expressed as a percentage of the portfolio volatility. Notice that the Euler's theorem implies $\sum_{i=1}^{n} S_i(\omega, \Sigma) = 1$. The optimal ERC portfolio ω^{ERC} equalizes the risk contribution of all assets, such as

$$S_i(\omega^{ERC}, \Sigma) = \frac{1}{n} \;\; \forall\, i = 1, ..., n. \tag{8}$$

This condition corresponds to the FOC of the general risk-based allocation program (Equation 1) with $\gamma = 1$ and $\delta = 0$.

Except under very restrictive conditions ($n = 2$ or equal correlations, see Roncalli (2014) for more details), there is no closed-form solution for ω^{ERC}. In the general case, the ERC portfolio is the numerical solution of the following quadratic optimization program:

$$\omega^{ERC} = \arg\min_{\omega} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(S_i(\omega,\Sigma) - S_j(\omega,\Sigma)\right)^2 \quad \text{u.c.} \begin{cases} e'\omega = 1 \\ \omega \geq 0 \end{cases} \tag{9}$$

Maillard et al. (2010) show that the MV, ERC and Equally Weighted (EW) portfolio volatilities can be ranked in the following order $\sigma_{MV} \leq \sigma_{ERC} \leq \sigma_{EW}$. Hence, the ERC portfolio is naturally located between MV and EW and thus appears as good potential substitutes for these traditional approaches.

2.3 Maximum diversification

The MD strategy, introduced by Choueifaty and Coignard (2008), also aims at diversifying the portfolio. The optimal MD weights are determinedin order to maximize the so-called diversification ratio $DR(\omega;\Sigma)$, defined as the portfolio's weighted average asset volatility to its actual volatility.

$$DR(\omega;\Sigma) = \frac{\omega'\sigma}{(\omega'\Sigma\omega)^{1/2}} \tag{10}$$

with $\sigma = (\sigma_1, ..., \sigma_n)'$ the vector of individual volatilities. This ratio can be interpreted as the ratio between undiversified and diversified volatility of the portfolio (Choueifaty and Coignard (2008)). Formally, the optimal portfolio ω^*_{MD} is obtained by resolving the following optimization program:

$$\omega^{MD} = \arg\max_{\omega} DR(\omega;\Sigma) \quad \text{u.c.} \begin{cases} e'\omega = 1 \\ \omega \geq 0 \end{cases} \tag{11}$$

Notice that the FOC of the MD program are equivalent to those of the general risk-based allocation program for $\gamma = 0$ and $\delta = 1$.

3 Benchmarking method

In this section, we detail the implementation of the conditional and unconditional risk-based strategies and present the related estimation approaches. Then, we present our comparison framework and the criteria used to assess the out-of-sample performance of each strategy.

3.1 Conditional and unconditional approaches

In order to compare the conditional and unconditional approaches, we consider a sample $\{r_1, ..., r_T\}$ of historical asset returns. By definition, risk-based strategies depend on the covariance matrix.[5] Under the stationarity assumption for asset returns, the unconditional covariance matrix is constant over time, meaning that $\Sigma_u = \mathbb{V}(r_t), \forall t = 1, ..., T$. As a consequence, the optimal portfolio weights $\omega_u^* = \lambda(\Sigma_u; \delta, \gamma)$ are also constant for $t = 1, ..., T$ and there is no turnover. This theoretical property clearly illustrates the main advantage of the unconditional approach for risk-based investment. However, such an approach assumes that the volatilities and correlations are constant whatever the economic and financial conditions (crisis period or not, news, etc.). On the contrary, the use of a conditional risk measure $\Sigma_{c,t} \equiv \mathbb{V}(r_t|\mathcal{F}_{t-h})$ implies a positive turnover, since the corresponding optimal weights $\omega_{c,t}^* = \lambda(\Sigma_{c,t}; \delta, \gamma)$ are time-varying with the information set $\mathcal{F}_{t-h}$. Thus, the conditional approach allows to incoporate the latest information in the optimal allocation. This is of great interest within a changing environment. For instance, Martellini, Milhau and Tarelli (2014) evoke the case of a risk parity strategy used for asset allocation in a context of low bond yield environment. This strategy will inevitably involve a substantial overweighting of bonds with respect to equities. Such an allocation may be problematic when a drop in long-term bond prices is likely to occur. Another difference between

[5]Other risk measures can be considered here. For instance, Martellini et al. (2014) define a new class of conditional risk parity portfolios with respect to downside risk measures such as semi-variance, Value-at- Risk (VaR) or expected shortfall.

unconditional and conditional approaches, is that the latter necessarily implies the use of a model for the conditional covariance matrix, and hence raises the issue of misspecification error.

In practice, the covariance matrices are unobservable and have to be estimated. Assuming that returns are independently and identically distributed (i.i.d), the unconditional covariance matrix can be estimated by its empirical counterpart, i.e. the sample covariance matrix of asset returns defined as

$$\hat{\Sigma}_u = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})(r_t - \bar{r})'. \tag{12}$$

However, the i.i.d. assumption is clearly violated. First, volatility of asset returns tends to change over time, and periods of high and low volatility tend to cluster together. Second, it is well known that in crisis periods the correlations between assets tend to sharply increase, reducing the diversification opportunities (Longin and Solnik (2001)). Besides, the sample covariance estimator gives the same weight to each of the past observations. So, practitioners and academics generally regularly rebalance their portfolios and re-estimate the unconditional covariance matrix by using the most recent observations, with a rolling window estimation approach.

Let us assume that the investor rebalances his portfolio every H periods and keeps the portfolio weights constant between two rebalancing dates. At each rebalancing date s, the investor forecasts the covariance matrix of the cumulated returns $R_{s+1:s+H}$ over the period $s+1$ to $s+H$. The out-of-sample forecast of the unconditional covariance matrix is simply defined as the sample covariance matrix of past asset returns. As usual in the literature, we consider a rolling-window estimator based on M observations of the returns r_t, for $t = s-M, .., s$. As displayed in Figure (1), the rebalancing dates are then set at $s = M$, $M+H$, $M+2H, ..., T-M-H$. For each rebalancing date s, the out-of-sample forecast of covariance matrix for the

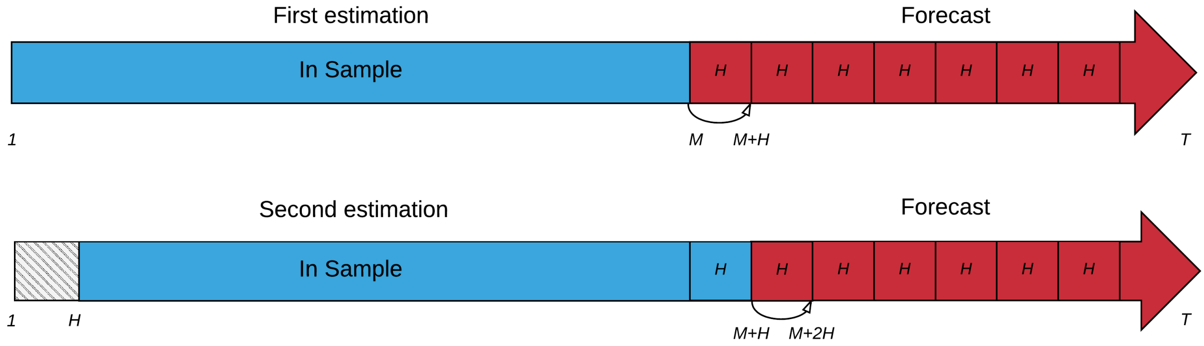

Figure 1: Rolling-window framework

cumulated returns $R_{s+1:s+H}$ is given by

$$\hat{\Sigma}_{u,s+1:s+H} = \frac{H}{M-1}\sum_{j=1}^{M}(r_{s-j+1} - \bar{r}_s)(r_{s-j+1} - \bar{r}_s)', \ \forall s \tag{13}$$

with $\bar{r}_s = M^{-1}\sum_{j=1}^{M} r_{s-j+1}$.[6] Then, the forecasted covariance matrix $\hat{\Sigma}_{u,s+1:s+H}$ is used to determine the optimal portfolio weights $\omega^*_{u,t} = \lambda\left(\hat{\Sigma}_{u,s+1:s+H}; \delta, \gamma\right)$, which are fixed over the periods $t = s+1, ..., s+H$.

We proceed in the same way for the conditional portfolios $\omega^*_{c,t}$. The only difference is that this approach implies the use of a dynamic model for the conditional covariance matrix $\Sigma_{c,s} = \mathbb{V}(R_{s+1:s+H}|\mathcal{F}_s)$. Many alternative multivariate GARCH type-models can be considered here (see Bauwens et al. (2006) for a survey). Here, we consider the Dynamic Conditional Correlation (DCC) model introduced by Engle (2002) which is the most used in the literature. For ease of presentation, we consider the case $H = 1$ and $R_{s+1:s+H} = r_{s+1}$. Formally, at each rebalancing date $s = t$ we assume that

$$\Sigma_{c,t} = D_t C_t D_t \tag{14}$$

[6] An alternative estimator for $\Sigma_{u,s+1:s+H}$ is given by the sample covariance matrix of the past *cumulated* returns

$$\hat{\Sigma}_{u,s+1:s+H} = \frac{1}{M/H-1}\sum_{j=1}^{M/H}(R_{s-j+1:s-j+H} - \overline{R}_s)(R_{s-j+1:s-j+H} - \overline{R}_s)' \qquad \text{TO BE DONE}$$

Here, we only consider the estimator given by Equation (13), which is based on the past *daily* returns, since its precision (i.e. the number of observations used to compute the sample covariance) does not change with the rebalancing horizon H.

where C_t denotes the correlation matrix and D_t is a diagonal matrix defined as

$$D_t = diag\{\sigma_{i,t}\}. \tag{15}$$

The conditional variance for the i^{th} asset return, denoted $\sigma_{i,t}^2$, follows a univariate GARCH(1,1) process. Let $\varepsilon_t = D_t^{-1} r_t$ be the standardized returns and define Q_t a symmetric positive definite matrix such that

$$C_t = (diag\{Q_t\})^{-1/2} \; Q_t \; (diag\{Q_t\})^{-1/2}. \tag{16}$$

The dynamics of Q_t is given by the following GARCH-type expression:

$$Q_t = \bar{Q}\,(1-\alpha-\beta) + \alpha\left(\varepsilon_{t-1}\varepsilon_{t-1}'\right) + \beta Q_{t-1} \tag{17}$$

where $\bar{Q}$ is the unconditional correlation matrix of the standardized returns ε_t, α and β are two positive parameters with $\alpha + \beta < 1$. At each rebalancing date s, the model parameters $\theta = (\alpha, \beta)'$ are estimated by quasi-maximum likelihood (QML). As for the unconditional case, we consider a rolling window estimator based on M past returns r_t for $t = s - M, .., s$. The conditional covariance matrix forecast $\widehat{\Sigma}_{c,s+1:s+H} = \widehat{\mathbb{V}}(R_{s+1:s+H}|\mathcal{F}_s)$ is then used to compute the optimal allocation $\omega_{c,t}^* = \lambda\left(\widehat{\Sigma}_{c,s+1:s+H}; \delta, \gamma\right)$ for $t = s+1, ..., s+H$.[7]

This process is repeated for the whole period. Finally, we get $T - M$ out-of-sample (ex-post) portfolio returns $\omega_{u,t}^{*\prime} r_t$ and $\omega_{c,t}^{*\prime} r_t$, with $t = M, ..., T$, for both unconditional and conditional approaches of the risk-based strategies MV, ERC, and MD.

[7]Various methods can be used to compute the forecast $\widehat{\Sigma}_{c,s+1:s+H}$, including dynamic forecast, simulation, or approximation. As for the unconditional case, we consider the approximation $\widehat{\Sigma}_{c,s+1:s+H} = H \times \widehat{\mathbb{V}}(r_{s+1}|\mathcal{F}_s)$.

3.2 Performance criteria

The objective is to compare the out-of-sample performances of the unconditional and conditional approaches for each of the three risk-based strategies. For this purpose, we use the same performance criteria as in DeMiguel et al. (2009), namely (1) the Sharpe ratio, (2) the return-loss, (3) the turnover, and (4) the portfolio variance. The rest of this section briefly presents these 5 criteria. For ease of presentation, we do not index the returns and the standard deviation for each strategy and each approach.

The out-of-sample Sharpe ratio (SR thereafter) represents the expected return by unit of risk and is computed as

$$SR = \frac{\hat{\mu}}{\hat{\sigma}} \tag{18}$$

with $\hat{\mu}$ the empirical mean and $\hat{\sigma}$ the standard deviation of the (ex-post) returns of the risk-based portfolio.[8] For each risk-based investment strategy, we also compute the p-value associated to the test of the null hypothesis of no-difference between the Sharpe ratios of unconditional and conditional approaches. To do so, we use the test methodology used by DeMiguel et al. (2009), initialy introduced by Jobson and Korkie (1981) with the correction by Memmel (2003) [9].

Second, we consider the return-loss (RL thereafter) of the unconditional approach with re-

[8] Formally, for each approach (conditional and unconditional), indexed by $z = \{u, c\}$, and each risk-based investment strategy, indexed by $g = \{MV, ERC, MD\}$, we have

$$\widehat{\mu} = \frac{1}{T-M} \sum_{t=1}^{T-M} \left(\omega_{z,t}^{g}\right)' r_t$$

$$\widehat{\sigma}^2 = \frac{1}{T-M-1} \sum_{t=1}^{T-M} \left(\left(\omega_{z,t}^{g}\right)' r_t - \widehat{\mu}\right)^2$$

[9] Formally, given two portfolios i and j, with $\hat{\mu}_i$, $\hat{\mu}_j$, $\hat{\sigma}_i$, $\hat{\sigma}_j$ and $\hat{\sigma}_{i,j}$ their estimated mean, variance and covariance over a sample of size $T - M$, the test of the hypothesis H_0: $\hat{\mu}_i/\hat{\sigma}_i - \hat{\mu}_j/\hat{\sigma}_j = 0$ is obtained by the following test statistic, which is asymptotically distributed as a standard normal:

$$\hat{z} = \frac{\hat{\sigma}_j\hat{\mu}_i - \hat{\sigma}_i\hat{\mu}_j}{\sqrt{\hat{\theta}}}, \text{ with } \hat{\theta} = \frac{1}{T-M}\left(2\hat{\sigma}_i^2\hat{\sigma}_j^2 - 2\hat{\sigma}_i\hat{\sigma}_j\hat{\sigma}_{i,j} + \frac{1}{2}\hat{\mu}_i^2\hat{\sigma}_j^2 + \frac{1}{2}\hat{\mu}_j^2\hat{\sigma}_i^2 - \frac{\hat{\mu}_i\hat{\mu}_j}{\hat{\sigma}_i\hat{\sigma}_j}\hat{\sigma}_{i,j}^2\right). \tag{19}$$

spect to the conditional approach. This criterium represents the additional return needed for the unconditional approach to perform as well as the conditional one. The RL is defined as

$$RL = \frac{\hat{\mu}_c}{\hat{\sigma}_c} \times \hat{\sigma}_u - \hat{\mu}_u \tag{20}$$

with $\hat{\mu}_c$, $\hat{\sigma}_c$, $\hat{\mu}_u$ and $\hat{\sigma}_u$ respectively denote the empirical mean and standard deviation of the conditional and unconditional approaches (ex-post) returns.

Finally, we compare the turnover of the conditional and unconditional approaches. For that, many indicators can be used. Here, we measure the average turnover by

$$\text{Turnover} = \frac{1}{S} \sum_{s=1}^{S} \sum_{i=1}^{n} \left| \omega_{i,M+(s+1)H} - \omega_{i,M+SH} \right| \tag{21}$$

with $\omega_{i,s}$ the weight of i^{th} asset at rebalancing date s and $S = (T - M - H)/H$ the total number of rebalancing dates.

4 Data

In order to apply our benchmarking method, we consider 4 empirical datasets used by DeMiguel et al. (2009) to compare the out-of-sample performance of the naive portfolio to the sample-based mean-variance model. We extend their datasets to the end of 2016. Notice that DeMiguel et al. (2009) consider monthly returns. Instead, we consider daily returns as it is well-known that it is the most favorable sampling frequency to identify conditional heteroskedasticity, i.e. ARCH effects. Hence, we consider the most favorable framework to identify the gains of the conditional risk-based investment strategies.

These 4 datasets include various number of assets and types of exposure: (1) Industry, (2) International, (3) Market - Small minus Big - High minus Low (MKT/SMB/HML), and

(4) Size and Book-to-Market. The portfolios included in these 4 datasets are exactly similar to those considered by DeMiguel et al. (2009). In what follows, the value-weighted portfolios are constructed at the end of each June using the June market equity and NYSE breakpoints. A value weighted portfolio compute assets' weight based on its absolute and relative value as compared to other stocks in the portfolio.

Industry. The dataset "Industry" is composed of daily value-weighted returns on 10 industries ($n = 10$) in the United Sates. Formally, each NYSE, AMEX and NASDAQ stock is assigned to an industry. Industries considered are: Consumer-Discretionary, Consumer-Staples, Manufacturing, Energy, High-Technology, Telecommunication, Wholesale and Retail, Health, Utilities and Others. The portfolios are constructed at the end of June of each year. The portfolios' daily returns from January 2, 1996 to December 30, 2016 are taken from Kenneth French's website.

International. The dataset "International" contains $n = 8$ international equity indices from MSCI (Morgan Stanley Capital International) for Canada, France, Germany, Italy, Japan, Switzerland, the UK and the US. Data range from June 1, 1994 to October 6, 2016.

Market - Small minus Big - High minus Low. The "MKT-SMB-HML" dataset represents 3 portfolios ($n = 3$) from Fama/French factors. Data consists in firms listed on the NYSE, AMEX and NASDAQ. The first portfolio is "Market" (MKT), which is the value-weighted excess return over the one-month Treasury bill rate on the US equity market. Then "Small minus Big" (SMB) is a zero-cost portfolio which is long in small-cap stocks and short in large-cap stocks. Formally SMB is the average return on the three small portfolios minus the average return on the three big portfolios. Finally, "High minus Low"(HML) is a zero-cost portfolio which is long in high book-to-market stocks and short in low book-to-market stocks. It represents the average return on the two value portfolios minus the average return on the two growth portfolios. Data from January 2, 1996 to December 30, 2016 are taken from Kenneth French's website.

Size and Book-to-Market. The SMB dataset is composed of twenty porfolios ($n = 20$) sorted by size and book-to-market ratio. As in DeMiguel et al. (2009) we exclude the five portfolio containing the largest firms. This results in the intersection of 4 portfolios formed on size (market equity) and 5 portfolios formed on the book-to-market ratio (book equity over market equity). Portfolios are constructed at the end of each June. Data from January 2, 1996 to December 30, 2016 are taken from Kenneth French's website.

5 Empirical results

In this section, we compare the different criteria for each risk-based strategy on unconditional and conditional frameworks and for each database. We also consider the naive strategy (or equally weighted portfolio) as a benchmark. This strategy involves holding a portfolio weight $\omega = 1/n$ in each of the n assets (DeMiguel et al., 2009). For each strategy, except for the naive one, we consider different rebalancing frequencies: $H = 1$ for daily, $H = 5$ for weekly, and $H = 22$ for monthly rebalancing, respectively. We also consider two rolling window sizes, $M = 500$ and $M = 1000$, in order to assess the influence of the estimation sample size on the out-of-sample performances of the portfolios. The expected outcome of the changes in those parameters depends on two opposite effects.

First, increasing the estimation window size diminishes the weight of recent information, and makes the covariance out-of-sample estimates less reactive to new information. At the limit, they are constant over time and converge to the true unconditional matrix. Under mild regularity assumptions, as soon as $M \to \infty$ with $M/n \to k > 0$, we get for $H = 1$:

$$\underset{M\to\infty}{\operatorname{p\,lim}} \hat{\Sigma}_{u,s+1} = \Sigma_u = \mathbb{V}(r_t) \tag{22}$$

$$\underset{M\to\infty}{\operatorname{p\,lim}} \hat{\Sigma}_{c,s+1}(\widehat{\theta}) = \Sigma_{c,t+1}(\theta_0) = \mathbb{V}(r_t|\mathcal{F}_{t-1}) \tag{23}$$

with θ_0, the true value of the MGARCH model. So, the optimal unconditional portfolios tend to be time invariant, i.e. $\omega^*_{u,t} = \lambda\left(\hat{\Sigma}_{u,s+1}; \delta, \gamma\right) \to \omega^*_u \lambda\left(\Sigma_u; \delta, \gamma\right)$ as $M \to \infty$, since the new H observations added at each rebalancing date do not change the information set used for the covariance estimation matrix. Obviously, this is not the case for the conditional portfolios, as $\mathbb{V}(r_t|\mathcal{F}_{t-1})$ changes with information available to date. Consequently, over a long evaluation period with financial crises and recoveries, we would expect that the relative performances of the conditional portfolios will improve (relatively to those of the unconditional portfolios) as the sample size M increases.

Second, the influence of the rebalancing frequency H on the relative out-of-sample performances of the conditional approach is less clear. For simplicity, let us denote by $\hat{\Sigma}_{u,s+H}$ the covariance matrix of the daily (instead of cumulated) returns at horizon H. Consider the special case in which the sample size M tends to infinity, implying that $\hat{\Sigma}_{c,s+H}(\widehat{\theta})$ converges to $\Sigma_{c,s+H}(\theta_0)$. In this case, when the rebalancing horizon H tends to infinity, $\hat{\Sigma}_{u,s+H}$ converges to the unconditional covariance matrix, since under stationarity assumptions, we have

$$\lim_{H\to\infty} \Sigma_{c,s+H}(\theta_0) = \Sigma_u \tag{24}$$

Consequently, when M and H tend to infinity, we have

$$\underset{H,M\to\infty}{\operatorname{p\,lim}} \hat{\Sigma}_{c,s+H}(\widehat{\theta}) = \underset{H,M\to\infty}{\operatorname{p\,lim}} \hat{\Sigma}_{u,s+H} = \Sigma_u \tag{25}$$

Under these assumptions, the unconditional and conditional risk-based optimal portfolios tend to be equivalent, i.e. $\omega^*_{u,t} \to \omega^*_{c,t}$, as $H \to \infty$. Even these results cannot be extended for finite sample sizes M, we would expect that the differences in the out-of-sample performances of both portfolios tend to decrease with M.

Table 5 reports the results for the out-of-sample SR for each rebalancing frequency H and sample size M. It also displays the p-value of the difference between conditional and unconditional portfolios. P-values in bold indicate significant differences between SRs. The main takeaway is that the conditional approach does not improve the performance of the portfolio in terms of SR and even worse, it can deteriorate it. Most of the SR differences are not significant at the 5 or 10% level. Furthermore, within the rare cases for which the differences are significant, the SRs of the unconditional strategies are generally higher than those of the conditional ones. For instance, for the portfolio MKT-SMB-HML with $M = 500$, the SR differences observed for all the strategies are not significant, except for the MV and MD with $H = 1$. But in these two cases, the SR of the unconditional strategies are largely higher than those of the conditional ones: 0.0309 versus 0.0133 for the MV strategy, 0.0327 versus 0.0220 for the MD strategy. As expected, the SRs of the unconditional approaches are stable with the rebalancing frequency H, but we observe an improvement for the SRs of the conditional strategies. Hence, the performance differences of both approaches tend to decrease with H. For instance, for the portfolio MKT-SMB-HML with $M = 500$, the SR difference of the MV strategy decreases from 0.2957 to 0.008 when H increases from $H = 1$ to $H = 22$. Besides, the SR differences are less and less significant with H. However, as expected, the gap between the SRs of the two approaches grows globally with the sample size M. For instance, the SR difference between the unconditional and conditional RP of the Size portfolio for $H = 1$, becomes significant for $M = 1,000$, while it was not in the case with $M = 500$. Notice that this is the only case where the conditional performs better than the unconditional approach. We can also notice that SRs of both approaches improves with M, which is the logical consequence of the reduction of the estimation errors.

Table 2 shows the results for the return-loss (RL), i.e. the additional return needed for the unconditional approach to perform as well as the conditional one. Recall that when RL is negative, the unconditional approach performs better than the conditional one. We report the results for each rebalancing frequency H and sample size M. The conclusions here confirm those obtained with the SR. Our main result is that, whether the RL positive or negative, the gains or losses in terms of returns are negligible. In fact, whatever the portfolio, the rebalancing frequency, and the estimation window size, the RL is always close to 0. For the portfolio "Size", the conditional approach dominates the unconditional approach, whereas the unconditional seems to be better for the portfolio "International". The RL is generally decreasing with M. For instance, for the portfolio Size, with $H = 22$ for the MV strategy, the RL is going from 0.0033 with $M = 500$ to -0.0041 with $M = 1000$, many positive RL with $M = 500$ become negative when $M = 1000$. Finally, there is no clear-cut conclusion as regards the relationship between the rebalancing horizon H and the out-of-sample gains in terms of returns.

As expected, the unconditional approach has a lower turnover than the conditional one. Table 5 reports the turnover for each strategy, each rebalancing date and each M. We also display the gap between unconditional and conditional turnovers. We can observe that the turnover of the conditional approach is always higher than that of the unconditional one. For all the strategies, the turnover decreases with H for the conditional approach, going for instance from 0.3743 for daily rebalancing to 0.0521 for monthly rebalancing for the MV strategy applied to the portfolio "Size", with $M = 500$. The turnover of the unconditional approach is logically more stable with H, which leads to a decreasing gap. Then, increasing the rebalancing horizon H benefits to the conditional approach, while the estimation sample size M shows no effect on the gaps between conditional and unconditional turnovers but a general reduction of both.

The main improvement due to the conditional strategy only concerns ex-post risk. Table 5 reports the empirical variance of the out-of-sample portfolio returns. For each strategy, we also display the difference between the variances of the unconditional and conditional portfolios. Values in bold correspond to the cases where the variance of the conditional approach is lower than that of the unconditional one. In most cases, allocation strategies with conditional approach have the lowest ex-post variance. This result comes from the higher reactiveness of the conditional portfolios. The gap between variances of unconditional and conditional approaches generally decreases with H. Indeed, if both variances increase with the rebalancing frequency, the volatility increase for the conditional portfolios is faster. Notice that the variance is increasing with M for each approach and each strategy, but the gap grows between both approaches.

To sum up, the implementation of a conditional approach shows an improvement of the risk dimension with a decline in the variance, while the performance is not enhanced and the turnover is largely increased.

Table 1: Sharpe Ratio

Sharpe Ratio M=500

	Industry			MKT-SMB-HML			International			Size		
	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22
Uncond RP	0.0354	0.0355	0.0356	0.0114	0.0242	0.0108	0.0227	0.0227	0.0229	0.0366	0.0368	0.0370
Cond RP	0.0350	0.0353	0.0357	0.0168	0.0173	0.0163	0.0201	0.0220	0.0222	0.0371	0.0370	0.0374
P-value	*0.5780*	*0.8906*	*0.9614*	*0.5517*	*0.6254*	0.8939	***0.0512***	*0.7948*	*0.8965*	*0.3022*	*0.8398*	*0.8181*
Uncond MV	0.0431	0.0433	0.0429	0.0309	0.0313	0.0330	0.0285	0.0291	0.0293	0.0466	0.0466	0.0464
Cond MV	0.0430	0.0436	0.0423	0.0133	0.0179	0.0250	0.0168	0.0226	0.0209	0.0534	0.0487	0.0492
P-value	*0.9857*	*0.9163*	*0.9678*	***0.0000***	*0.1199*	0.6471	***0.0005***	*0.3740*	*0.5772*	*0.0597*	*0.9309*	*0.8989*
Uncond MD	0.0344	0.0346	0.0347	0.0327	0.0331	0.0342	0.0239	0.0239	0.0238	0.0313	0.0314	0.0319
Cond MD	0.0348	0.0350	0.0348	0.0220	0.0247	0.0297	0.0206	0.0222	0.0231	0.0320	0.0315	0.0321
P-value	*0.4569*	*0.7160*	*0.8452*	***0.0000***	*0.1160*	0.6701	***0.0174***	*0.5262*	*0.8594*	*0.8740*	*0.7947*	*0.9395*
EW	0.0335	0.0336	0.0336	0.0315	0.0317	0.0318	0.0209	0.0208	0.0211	0.0347	0.0349	0.0350

Sharpe Ratio M=1000

	Industry			MKT-SMB-HML			International			Size		
	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22
Uncond RP	0.0325	0.0327	0.0320	0.0260	0.0221	0.0402	0.0168	0.0168	0.0165	0.0365	0.0367	0.0360
Cond RP	0.0320	0.0319	0.0309	0.0221	0.0186	0.0278	0.0137	0.0154	0.0157	0.0373	0.0373	0.0364
P-value	*0.5323*	*0.6556*	*0.7766*	*0.5417*	*0.8773*	*0.7238*	***0.0352***	0.6539	*0.8887*	***0.0636***	*0.5432*	*0.8669*
Uncond MV	0.0443	0.0444	0.0436	0.0380	0.0383	0.0373	0.0238	0.0242	0.0239	0.0476	0.0478	0.0469
Cond MV	0.0419	0.0413	0.0410	0.0210	0.0260	0.0266	0.0098	0.0145	0.0153	0.0516	0.0474	0.0435
P-value	*0.3601*	*0.6166*	*0.8429*	***0.0002***	*0.2094*	*0.6113*	***0.0003***	0.2498	*0.5829*	*0.6421*	*0.6596*	*0.7223*
Uncond MD	0.0322	0.0324	0.0317	0.0397	0.0400	0.0386	0.0188	0.0189	0.0184	0.0293	0.0295	0.0288
Cond MD	0.0319	0.0313	0.0311	0.0292	0.0324	0.0310	0.0144	0.0162	0.0173	0.0305	0.0302	0.0295
P-value	*0.2598*	*0.4771*	*0.7828*	***0.0003***	*0.2064*	*0.5511*	***0.0072***	0.4364	*0.8393*	*0.7189*	*0.9841*	*0.9976*
EW	0.0297	0.0299	0.0292	0.0353	0.0355	0.0347	0.0154	0.0152	0.0150	0.0346	0.0349	0.0342

For each empirical dataset, this table reports the sharpe ratio for each strategy for the conditional and unconditional approach and for each H and M. It also displays the p-value of the difference between unconditional and conditional sharpe ratios. P-values are in bold when the difference is statistically significant.

Table 2: Return-loss

Return-loss M=500	Industry			MKT-SMB-HML			International			Size		
	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22
RP	-0.0004	-0.0002	**0.0002**	**0.0028**	-0.0037	**0.0029**	-0.0025	-0.0008	-0.0007	**0.0006**	**0.0002**	**0.0005**
MV	-0.0001	**0.0002**	-0.0006	-0.0063	-0.0048	-0.0029	-0.0100	-0.0056	-0.0072	**0.0079**	**0.0025**	**0.0033**
MD	**0.0005**	**0.0004**	**0.0001**	-0.0040	-0.0032	-0.0017	-0.0029	-0.0015	-0.0006	**0.0010**	**0.0002**	**0.0004**

Return-loss M=1000	Industry			MKT-SMB-HML			International			Size		
	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22
RP	-0.0006	-0.0009	-0.0012	-0.0020	-0.0020	-0.0053	-0.0032	-0.0015	-0.0009	**0.0012**	**0.0008**	**0.0004**
MV	-0.0022	-0.0029	-0.0024	-0.0065	-0.0048	-0.0042	-0.0125	-0.0086	-0.0077	**0.0049**	-0.0005	-0.0041
MD	-0.0004	-0.0012	-0.0007	-0.0043	-0.0031	-0.0032	-0.0041	-0.0024	-0.0010	**0.0017**	**0.0011**	**0.0010**

For each empirical dataset, this table reports the return-loss for the unconditional approach and for each H and M. The RL is in bold when the conditional approach performs better (i.e. when it is positive).

Table 3: Turnover

Turnover M=500

	Industry			MKT-SMB-HML			International			Size		
	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22
Uncond RP	0.0014	0.0039	0.0112	0.0079	0.0116	0.0369	0.0019	0.0052	0.0143	0.0008	0.0026	0.0085
Cond RP	0.0317	0.0670	0.1050	0.0977	0.1434	0.2634	0.0348	0.0753	0.1249	0.0253	0.0545	0.0792
Gap	*-0.0303*	*-0.0631*	*-0.0938*	*-0.0898*	*-0.1318*	*-0.2265*	*-0.0329*	*-0.0701*	*-0.1106*	*-0.0245*	*-0.0519*	*-0.0707*
Uncond MV	0.0100	0.0269	0.0716	0.0025	0.0076	0.0229	0.0100	0.0268	0.0716	0.0163	0.0480	0.1431
Cond MV	0.2296	0.4854	0.6846	0.0514	0.1140	0.1938	0.1601	0.3502	0.5747	0.3743	0.7928	1.1465
Gap	*-0.2196*	*-0.4585*	*-0.6130*	*-0.0489*	*-0.1063*	*-0.1708*	*-0.1502*	*-0.3234*	*-0.5031*	*-0.3580*	*-0.7448*	*-1.0034*
Uncond MD	0.0113	0.0310	0.0793	0.0021	0.0062	0.0184	0.0090	0.0237	0.0549	0.0184	0.0497	0.1292
Cond MD	0.0743	0.1630	0.2778	0.0364	0.0805	0.1357	0.0579	0.1302	0.2257	0.0727	0.1727	0.3138
Gap	*-0.0630*	*-0.1320*	*-0.1985*	*-0.0343*	*-0.0743*	*-0.1173*	*-0.0489*	*-0.1065*	*-0.1708*	*-0.0543*	*-0.1230*	*-0.1845*
EW	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Turnover M=1,000

	Industry			MKT-SMB-HML			International			Size		
	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22
Uncond RP	0.0007	0.0021	0.0063	0.0031	0.0075	0.0302	0.0010	0.0027	0.0079	0.0004	0.0014	0.0049
Cond RP	0.0278	0.0630	0.1041	0.0881	0.1792	0.3087	0.0319	0.0720	0.1163	0.0217	0.0484	0.0766
Gap	*-0.0271*	*-0.0609*	*-0.0978*	*-0.0851*	*-0.1717*	*-0.2785*	*-0.0309*	*-0.0694*	*-0.1084*	*-0.0213*	*-0.0470*	*-0.0717*
Uncond MV	0.0055	0.0150	0.0425	0.0014	0.0047	0.0161	0.0052	0.0142	0.0401	0.0082	0.0257	0.0882
Cond MV	0.2131	0.4815	0.6966	0.0487	0.1157	0.2134	0.1447	0.3351	0.5341	0.3321	0.7273	1.0998
Gap	*-0.2076*	*-0.4665*	*-0.6541*	*-0.0474*	*-0.1111*	*-0.1973*	*-0.1395*	*-0.3209*	*-0.4940*	*-0.3240*	*-0.7016*	*-1.0117*
Uncond MD	0.0062	0.0171	0.0448	0.0012	0.0039	0.0131	0.0047	0.0125	0.0304	0.0105	0.0290	0.0814
Cond MD	0.0716	0.1749	0.3224	0.0340	0.0804	0.1522	0.0559	0.1334	0.2417	0.0752	0.1844	0.3657
Gap	*-0.0654*	*-0.1578*	*-0.2777*	*-0.0328*	*-0.0765*	*-0.1391*	*-0.0512*	*-0.1209*	*-0.2113*	*-0.0647*	*-0.1554*	*-0.2843*
EW	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

For each empirical dataset, this table reports the turnover for each strategy for the conditional and unconditional approach and for each H and M. It also displays the gap between unconditional and conditional turnovers. Gaps are in bold when the conditional approach performs better than the unconditional one (i.e. when the conditional turnover is smaller).

Table 4: Variance

Variance M=500

	Industry			MKT-SMB-HML			International			Size		
	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22
Uncond RP	1.2131	1.2140	1.2183	0.2795	0.2850	0.2825	0.9931	0.9938	0.9973	1.7722	1.7731	1.7776
Cond RP	1.1849	1.1874	1.1949	0.3227	0.2783	0.3058	0.9542	0.9491	0.9602	1.7397	1.7415	1.7466
Gap	***0.0281***	***0.0266***	***0.0234***	*-0.0431*	***0.0066***	*-0.0234*	***0.0389***	***0.0448***	***0.0371***	***0.0325***	***0.0316***	***0.0310***
Uncond MV	0.8086	0.8096	0.8151	0.1280	0.1291	0.1313	0.7323	0.7331	0.7406	1.3469	1.3516	1.3688
Cond MV	0.8147	0.8237	0.8209	0.1145	0.1193	0.1277	0.7119	0.7026	0.7117	1.2522	1.2600	1.2999
Gap	*-0.0061*	*-0.0141*	*-0.0058*	***0.0135***	***0.0098***	***0.0037***	***0.0205***	***0.0305***	***0.0289***	***0.0947***	***0.0916***	***0.0688***
Uncond MD	1.2877	1.2898	1.2938	0.1418	0.1426	0.1440	0.7985	0.7968	0.7991	1.8568	1.8594	1.8717
Cond MD	1.1888	1.1919	1.2041	0.1278	0.1300	0.1356	0.7651	0.7608	0.7674	1.7858	1.7917	1.8102
Gap	***0.0988***	***0.0979***	***0.0897***	***0.0141***	***0.0126***	***0.0084***	***0.0334***	***0.0359***	***0.0317***	***0.0710***	***0.0677***	***0.0615***
EW	1.3488	1.3492	1.3518	0.2636	0.2636	0.2640	1.1698	1.1710	1.1730	1.8437	1.8441	1.8472

Variance M=1000

	Industry			MKT-SMB-HML			International			Size		
	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22	H=1	H=5	H=22
Uncond RP	1.2572	1.2580	1.2632	0.2781	0.3082	0.1818	1.0802	1.0812	1.0834	1.8950	1.8960	1.9024
Cond RP	1.2215	1.2237	1.2290	0.2510	0.4653	0.1990	1.0197	1.0155	1.0282	1.8477	1.8490	1.8621
Gap	***0.0357***	***0.0343***	***0.0342***	***0.0272***	*-0.1571*	*-0.0173*	***0.0605***	***0.0657***	***0.0551***	***0.0473***	***0.0469***	***0.0403***
Uncond MV	0.8693	0.8702	0.8781	0.1475	0.1484	0.1508	0.7940	0.7950	0.8016	1.4877	1.4912	1.5043
Cond MV	0.8453	0.8505	0.8616	0.1229	0.1280	0.1325	0.7459	0.7372	0.7570	1.3393	1.3410	1.4164
Gap	***0.0240***	***0.0197***	***0.0166***	***0.0246***	***0.0205***	***0.0183***	***0.0481***	***0.0578***	***0.0446***	***0.1484***	***0.1501***	***0.0879***
Uncond MD	1.3568	1.3585	1.3637	0.1671	0.1680	0.1700	0.8441	0.8440	0.8489	2.0377	2.0395	2.0494
Cond MD	1.2442	1.2473	1.2537	0.1369	0.1394	0.1422	0.8026	0.8000	0.8158	1.9290	1.9339	1.9553
Gap	***0.1126***	***0.1112***	***0.1100***	***0.0301***	***0.0285***	***0.0278***	***0.0415***	***0.0441***	***0.0330***	***0.1087***	***0.1056***	***0.0942***
EW	1.3815	1.3819	1.3863	0.2864	0.2865	0.2871	1.2526	1.2541	1.2544	1.9527	1.9532	1.9583

For each empirical dataset, this table reports the variance for each strategy for the conditional and unconditional approach and for each H and M. It also displays the gap between unconditional and conditional variances. Gaps are in bold when the conditional approach performs better than the unconditional one (i.e. when the conditional variance is smaller).

6 Robustness check

In this section, we analyse the robustness of our results. For that, we apply the same methodology as in Section 3, but we use the Constant Conditional Correlation (CCC) model of Bollerslev (1990) instead of the DCC model of Engle (2002). The CCC model belongs to the same type of model as the DCC, namely the multivariate-GARCH type. The only difference is about the correlation matrix, which is supposed constant in the CCC model. Formally, at each rebalancing date, the covariance matrix of the CCC model $\tilde{\Sigma}_{c,t}$ is defined as:

$$\tilde{\Sigma}_{c,t} = D_t \bar{Q} D_t \tag{26}$$

with $\bar{Q}$ the unconditional correlation matrix of returns defined in Section 3.

Results are reported in Table 5 for $M = 500$ and in Table 6 for $M = 1000$. Note that we conduct our methodology for each of the 4 databases, however we only report the Sharpe ratios, the turnovers and the variance, because the return-losses haven't shown clear conclusions. We also report the results only for $H = 1$ and $H = 22$ in order to capture the largest evolutions.

Table 5 reports the results for $M = 500$, for each database and each strategy with daily and monthly rebalancing frequency. The gaps and p-values in bold are the cases where the differences between Sharpe ratios is significant (for the p-values) and where the conditional approach performs better (for the gaps). The results for the turnovers are similar to the ones with the DCC model. As for the DCCC model, the differences of the Sharpe ratios that are significant are the ones where the unconditional Sharpe ratio is higher. Regarding the turnover, we also observe a higher turnover for the conditional approach, as we expected. We observe a smaller improvement for the variance than with the DCC model. Indeed, the variance is sometimes higher for the unconditional strategy but the result is more mitigate.

In Table 6, we report the results for $M = 1000$. The differences between SRs remain unsignificant, except for the International database and for $H = 1$. The turnover is also higher for the conditional approach and for each strategy and rebalancing frequency. However, the variance is here always smaller for the conditional strategy. This could be explained by the fact that the CCC model is not as adjusted as the DCC to financial series; thus one needs a bigger estimation sample size to perform as well as the DCC model.

Table 5: CCC model, $M = 500$

CCC Model H=1; M=500

	Industry			MKT-SMB-HML			International			Size		
	SR	Turnover	Variance	SR	Turnover	Variance	SR	Turnover	Variance	SR	Turnover	Variance
Uncond RP	0.0354	0.0014	1.2131	0.0254	0.0067	0.5781	0.0227	0.0019	0.9931	0.0366	0.0008	1.7722
Cond RP	0.0353	0.0315	1.1759	0.0251	0.0336	0.1498	0.0197	0.0336	0.9667	0.0369	0.0252	1.7453
P-value/Gap	*0.9522*	*-0.0301*	***0.0371***	*0.9858*	*-0.0268*	***0.4283***	***0.0246***	*-0.0316*	***0.0264***	*0.5611*	*-0.0244*	***0.0269***
Uncond MV	0.0431	0.0100	0.8086	0.0309	0.0025	0.1280	0.0285	0.0100	0.7323	0.0466	0.0163	1.3469
Cond MV	0.0457	0.2164	0.8371	0.0163	0.0550	0.1393	0.0150	0.1825	0.7340	0.0521	0.3730	1.2653
P-value/Gap	*0.2899*	*-0.2064*	*-0.0284*	***0.0205***	*-0.0525*	*-0.0113*	***0.0006***	*-0.1725*	*-0.0016*	*0.1470*	*-0.3567*	***0.0816***
Uncond MD	0.0344	0.0113	1.2877	0.0327	0.0021	0.1418	0.0239	0.0090	0.7985	0.0313	0.0184	1.8568
Cond MD	0.0363	0.0334	1.1045	0.0248	0.0331	0.1443	0.0166	0.0347	0.8148	0.0308	0.0271	1.9083
P-value/Gap	*0.7662*	*-0.0221*	***0.1831***	***0.0404***	*-0.0310*	*-0.0024*	***0.0011***	*-0.0257*	*-0.0163*	*0.9788*	*-0.0087*	*-0.0515*
EW	0.0335	0.0000	1.3488	0.0315	0.0000	0.2636	0.0209	0.0000	1.1698	0.0347	0.0000	1.8437

CCC Model H=22; M=500

	Industry			MKT-SMB-HML			International			Size		
	SR	Turnover	Variance	SR	Turnover	Variance	SR	Turnover	Variance	SR	Turnover	Variance
Uncond RP	0.0356	0.0112	1.2183	0.0260	0.0369	0.2885	0.0229	0.0143	0.9973	0.0370	0.0085	1.7776
Cond RP	0.0358	0.1029	1.1851	0.0325	0.1343	0.1584	0.0216	0.1215	0.9688	0.0372	0.0788	1.7519
P-value/Gap	*0.9371*	*-0.0917*	***0.0333***	*0.8885*	*-0.0973*	***0.1301***	*0.8215*	*-0.1072*	***0.0285***	*0.8997*	*-0.0703*	***0.0257***
Uncond MV	0.0429	0.0716	0.8151	0.0330	0.0229	0.1313	0.0293	0.0716	0.7406	0.0464	0.1432	1.3687
Cond MV	0.0436	0.6404	0.8301	0.0263	0.1848	0.1508	0.0187	0.6403	0.7318	0.0505	1.1205	1.3161
P-value/Gap	*0.9392*	*-0.5688*	*-0.0150*	*0.8557*	*-0.1619*	*-0.0195*	*0.5552*	*-0.5687*	***0.0088***	*0.8021*	*-0.9774*	***0.0526***
Uncond MD	0.0347	0.0793	1.2938	0.0342	0.0184	0.1440	0.0238	0.0549	0.7991	0.0319	0.1292	1.8717
Cond MD	0.0372	0.1085	1.1117	0.0306	0.1149	0.1517	0.0191	0.1248	0.8201	0.0292	0.0888	1.9266
P-value/Gap	*0.9790*	*-0.0292*	***0.1821***	*0.8779*	*-0.0964*	*-0.0077*	*0.6584*	*-0.0699*	*-0.0211*	*0.6591*	***0.0404***	*-0.0550*
EW	0.0336	0.0000	1.3518	0.0318	0.0000	0.2640	0.0211	0.0000	1.1730	0.0350	0.0000	1.8472

For each empirical dataset, this table reports the criteria for each strategy for the conditional and unconditional approach and for $H = 1; 22$ and $M = 500$. It also displays the gap/p-value between unconditional and conditional criteria. Gaps/p-values are in bold when the conditional approach performs better than the unconditional one.

Table 6: CCC model, $M = 1,000$

CCC Model H=1; M=1,000

	Industry			MKT-SMB-HML			International			Size		
	SR	Turnover	Variance	SR	Turnover	Variance	SR	Turnover	Variance	SR	Turnover	Variance
Uncond RP	0.0325	0.0007	1.2572	0.0238	0.0015	0.2905	0.0168	0.0010	0.9931	0.0365	0.0004	1.8950
Cond RP	0.0325	0.0276	1.2142	0.0350	0.0298	0.1592	0.0142	0.0308	0.9667	0.0372	0.0216	1.8517
P-value/Gap	*0.9561*	*-0.0269*	***0.0430***	*0.2436*	*-0.0283*	***0.1313***	***0.0889***	*-0.0298*	***0.0264***	*0.1256*	*-0.0212*	***0.0433***
Uncond MV	0.0443	0.0055	0.8693	0.0380	0.0014	0.1475	0.0238	0.0052	0.7323	0.0476	0.0082	1.4877
Cond MV	0.0437	0.2061	0.8605	0.0322	0.0470	0.1400	0.0095	0.1502	0.7340	0.0518	0.3264	1.3468
P-value/Gap	*0.8287*	*-0.2006*	***0.0088***	*0.2873*	*-0.0456*	***0.0075***	***0.0006***	*-0.1450*	*-0.0016*	*0.5574*	*-0.3183*	***0.1410***
Uncond MD	0.0322	0.0062	1.3568	0.0397	0.0012	0.1671	0.0188	0.0047	0.7985	0.0293	0.0105	2.0377
Cond MD	0.0343	0.0301	1.1888	0.0366	0.0291	0.1498	0.0140	0.0329	0.8148	0.0287	0.0237	2.0062
P-value/Gap	*0.9848*	*-0.0239*	***0.1680***	*0.1822*	*-0.0279*	***0.0173***	***0.0259***	*-0.0282*	*-0.0163*	*0.3682*	*-0.0132*	***0.0315***
EW	0.0297	0.0000	1.3815	0.0353	0.0000	0.2864	0.0154	0.0000	1.1698	0.0346	0.0000	1.9527

CCC Model H=22; M=1,000

	Industry			MKT-SMB-HML			International			Size		
	SR	Turnover	Variance	SR	Turnover	Variance	SR	Turnover	Variance	SR	Turnover	Variance
Uncond RP	0.0320	0.0063	1.2632	0.0460	0.0103	0.2094	0.0165	0.0079	1.0834	0.0360	0.0049	1.9024
Cond RP	0.0312	0.1012	1.2202	0.0375	0.1254	0.1623	0.0159	0.1112	1.0290	0.0362	0.0760	1.8659
P-value/Gap	*0.8385*	*-0.0950*	***0.0430***	*0.7943*	*-0.1150*	***0.0471***	*0.9206*	*-0.1033*	***0.0544***	*0.9331*	*-0.0711*	***0.0365***
Uncond MV	0.0436	0.0425	0.8781	0.0373	0.0161	0.1508	0.0239	0.0401	0.8016	0.0469	0.0882	1.5043
Cond MV	0.0437	0.6564	0.8632	0.0376	0.1970	0.1480	0.0168	0.5474	0.7577	0.0432	1.0727	1.4285
P-value/Gap	*0.9911*	*-0.6139*	***0.0149***	*0.9983*	*-0.1809*	***0.0027***	*0.6703*	*-0.5074*	***0.0439***	*0.7201*	*-0.9845*	***0.0758***
Uncond MD	0.0317	0.0448	1.3637	0.0386	0.0131	0.1700	0.0184	0.0304	0.8489	0.0288	0.0814	2.0494
Cond MD	0.0332	0.1085	1.1925	0.0390	0.1226	0.1533	0.0168	0.1212	0.8412	0.0266	0.0869	2.0233
P-value/Gap	*0.9411*	*-0.0637*	***0.1712***	*0.9292*	*-0.1095*	***0.0167***	*0.8547*	*-0.0908*	***0.0076***	*0.6343*	*-0.0055*	***0.0261***
EW	0.0292	0.0000	1.3863	0.0347	0.0000	0.2871	0.0150	0.0000	1.2544	0.0342	0.0000	1.9583

For each empirical dataset, this table reports the criteria for each strategy for the conditional and unconditional approach and for $H = 1; 22$ and $M = 1,000$. It also displays the gap/p-value between unconditional and conditional criteria. Gaps/p-values are in bold when the conditional approach performs better than the unconditional one.

7 Conclusion

In this paper, we have proposed a comparison of unconditional and conditional approaches for the estimation of the covariance matrix. We have compared the performance of these approaches within the framework of risk-based investing, adopting the DCC-GARCH model from Engle (2002) for the estimation of the conditional covariance matrix. We have implemented a rolling-window estimation scheme with different rebalancing horizons and different sizes of estimation samples on 4 empirical datasets and for 3 risk-based strategies, with the naive strategy for benchmark.

Our results show that adopting a conditional approach benefits the risk-based strategies in terms of risk: the variance of the portfolio returns is weaker for the conditional approach whatever the rebalancing frequency or the estimation sample length. However, the performance in terms of returns is not enhanced: the Sharpe ratio is higher for the unconditional approach. We only observe a deterioration of the unconditional Sharpe ratios while increasing the estimation windows lenght, which improves the relative performances of the conditional approach. Finally, as expected, we observe a large increase of the portfolio turnovers for the conditional approach.

These findings are consistent with the theoretical properties according to which the unconditional and conditional optimal portfolios tend to be equivalent when $H \to \infty$ as well as when $M \to \infty$. Indeed, the differences between the comparison criteria decrease when one or both parameters H and M increase.

Appendix

A Risk-based generalized strategies

Denote by MRC_i the following expression:

$$MRC_i = \frac{\partial \sigma_p}{\partial \omega_i} = \frac{(\Sigma\omega)_i}{\omega'\Sigma\omega}, \tag{27}$$

which represents the marginal risk contribution of asset i. The marginal risk contribution of asset i can be seen as the sensitivity of the portfolio total risk to a small change in the weight of asset i. It is straightforward to find the first order condition of the optimization program (1):

$$\frac{\omega_i^\gamma}{\sigma_i^\delta} \times MRC_i = \frac{\omega_j^\gamma}{\sigma_j^\delta} \times MRC_j = \tau \qquad \forall (i,j) = 1, ..., N \tag{28}$$

with

$$\sum_{i=1}^{N} \omega_i = 1$$

where τ is a positive target constraint which is not necessary known to solve the program. In Table 7, the way to obtain our different risk-based portfolios with respect to γ and δ from the program (1) is reported.

Table 7: Risk-based strategies

Portfolio	(γ,δ)	Strategy definition
ERC	(1,0)	$\omega_i MRC_i = \omega_j MRC_j$
MV	(0,0)	$MRC_i = MRC_j$
MD	(0,1)	$\sigma_i^{-1} MRC_i = \sigma_j^{-1} MRC_j \quad (\rho_{ip} = \rho_{jp})$
1/N	$(\infty$,0)	$\omega_i = \omega_j = N^{-1}$

B Existence and uniqueness

In order to discuss the existence and uniqueness of the optimal risk-based portfolios, we use an alternative expression of the optimization program suggested by Jurczenko et al. (2013):

$$\begin{aligned} \omega^* &= \arg\min_{\omega} \ \tfrac{1}{2}\omega'\Sigma\omega \\ \text{u.c.} & \sum_{i=1}^{n} \tfrac{\sigma_i^\delta(\omega_i^{1-\gamma}-1)}{(1-\gamma)} \geq c, \end{aligned} \tag{29}$$

where c is a constant that depends on the risk-based strategy. The associated Lagrangian function is:

$$L(\omega;\lambda) = \frac{1}{2}\omega'\Sigma\omega - \lambda\left\{\sum_{i=1}^{n}\frac{\sigma_i^{\delta}(\omega_i^{1-\gamma}-1)}{(1-\gamma)} - c\right\} \tag{30}$$

whith λ the associated Lagrange multiplier. The first order condition is then given by:

$$\frac{\partial L(\omega;\lambda)}{\partial\omega} = \Sigma\omega - \lambda\nu, \tag{31}$$

with $\nu = (\sigma_1^{\delta}/\omega_1^{\gamma}...\sigma_n^{\delta}/\omega_n^{\gamma})'$. For each asset, this condition can be rewritten as:

$$\frac{(\omega_i^*)^{\gamma}}{\sigma_i^{\delta}} \times (\Sigma\omega)_i = \lambda \qquad \forall i = 1,...,n \tag{32}$$

where $(\Sigma\omega)_i$ denotes the i^{th} row of the matrix $\Sigma\omega$.

Define $\sigma_p = (\omega'\Sigma\omega)^{1/2}$ the portfolio volatility and $\partial\sigma_p/\partial\omega_i = (\Sigma\omega)_i/\sigma_p$ the marginal risk contribution of asset i. Then, for each asset, the first order condition can be rewritten as:

$$\frac{(\omega_i^*)^{\gamma}}{\sigma_i^{\delta}} \times \frac{\partial\sigma_p}{\partial\omega_i} = \frac{\lambda}{\sigma_p} \qquad \forall i = 1,...,n. \tag{33}$$

The optimal solution leads to equalize the modified risk contributions. We see that the first order condition of (1) and (29) are the same, that proves the equivalence of the two optimization programs. The uniqueness of the solution is guaranteed by the convexity of the program. The computation of the second order conditions gives:

$$\frac{\partial^2 L(\omega;\lambda)}{\partial\omega^2} = \Sigma + \gamma\lambda\kappa \tag{34}$$

with $\kappa = (\sigma_1^{\delta}/\omega_1^{\gamma+1}...\sigma_n^{\delta}/\omega_n^{\gamma+1})'$. The solution exists and is unique as long as this expression is positive. As γ and δ are positive parameters, the solution exists as long as Σ is a definite-positive matrix. Under this assumption, all optimal risk-based portfolios considered in this paper (MV, ERC, MD, 1/N) will be uniquely defined.

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