Intermittent Discounting

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Abstract

A novel theory of time discounting is proposed in which future consumption is less valuable than present one due to waiting costs incurred in the interval. Waiting is intermittent as consumer's attention is periodically distracted away from future gratification. The model replicates three important features of intertemporal preferences: present bias, the propensity to prefer immediate gratification to future ones, decreasing impatience, a decrease of the rate at which an outcome is discounted as the time horizon gets longer and non-additive discounting, according to which a sequence of trade-offs in a subdivided interval leads to more overall discounting than a single trade-off over the whole interval.

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1 Introduction

Time preferences play a central role in many fundamental decisions including how much to consume and save, invest in education or engage in health preserving activities. Economic models of intertemporal choice assume that people prefer early to late gratifications but very few elicit the discounting mechanism behind. This paper proposes a novel theory of time discounting which starts from the observation that waiting for a reward requires some mental effort and induces psychological costs as people have to resist temptation and cope with some amount of frustration. The more delayed the gratification, the longer the waiting period and the less valuable future utility. In addition, introspection and casual observations indicate that people do not continuously experience a disutility of waiting as they do not spend all their time in waiting states. Human time alternates between periods of absorption in daily activities and periods of conscious reminding of delayed gratifications during which waiting costs are incurred. Those may be triggered by an exogenous or external event or a cue, like passing by a bakery's front window, watching a tv advertisement, speaking of a new model of cell-phone with a colleague or being offered a piece of double chocolate fudge cake at a friend âĂŹ s birthday. The frequency of reminders may be reinforced in case of biased attention toward temptation cues.[1] Reminding may also spontaneously occur when the image of a gratification springs to mind, or when a need is felt, out of boredom, discomfort, stress, hunger, thirst or craving.

When individuals experience intermittent reminding, preference for early vs late gratifications depend on two factors: how strong future utilities are depreciated compared to present utility in waiting periods and how many times waiting periods are expected to occur until consumption. Both dimensions raise expected waiting costs and undermine consumer's willingness to delay consumption. Existing theories overlook the second factor despite the two matter for a comprehensive theory of time discounting. For example, a typical question asked to people who suffer from addiction is "how many times a day do you think of ...". In less extreme situations, repeated exposure to temptation goods may lead consumers to indulge, which is routinely exploited by the advertising industry.

It is also well documented by the psychological literature. In the famous marshmallow experiment in which children were offered a choice between one treat immediately or two if they waited for a short period, whether children kept their attention on the immediate reward did matter (Mischel et al., 1972).[2] More recently, Hofmann et al. (2012) investigate with an experience sampling method how often desires in everyday life (like eating, sleeping or drinking) are felt and how often they are enacted or inhibited. They find that people who were the best

[1] For example, smokers have been found to display selective attention for smoking-related cues (Mogg et al., 2003), and heavy drinkers toward alcohol-related cues (e.g., Townshend and Duka, 2001). See also Bernheim and Rangel (2004) for a theoretical analysis.

[2] According to Mischel et al. (1972): "Children waited much longer for a preferred reward when they were distracted from the rewards than when they attended to them directly. (...) Overall, attentional and cognitive mechanisms which enhanced the salience of the rewards shortened the length of voluntary delay, while distractions from the rewards, overtly or cognitively, facilitated delay."

at self-control reported fewer temptations rather than better ability to resist temptations. Ent et al. (2015) also show that self-control is linked to avoiding, rather than merely resisting, temptation. Traditional theories of intertemporal choice have difficulties in accounting for those simple observations as pure time preferences are not properly distinguished from the frequency of temptations.

The implications of intermittent reminding for time preferences are investigated in several steps. I first pose a general multi-period setting in which an agent derives utility from a good which may be consumed now or latter. A disutility of waiting is also felt conditional on not having consumed the good yet and on reminding the good. Reminding is intermittent and random. In a second step, three axioms are posed, that preferences must satisfy. The first two guarantee a positive utility of consumption and a preference for early consumption, but only during reminding periods. The third axiom states that the consumer is indifferent between early and late consumption in non-reminding periods, i.e. when unrelated activities distract her away from future gratifications. I then pose a general model of intertemporal choice with waiting costs and show how preferences are constrained by the three axioms.

The resulting wait-based model of discounting replicates three important and robust features of intertemporal preferences: (*i*) present bias, the propensity to prefer immediate gratification to future ones, (*ii*) decreasing impatience (or hyperbolic discounting) according to which the rate at which an outcome is discounted over time decreases as the time horizon gets longer, (*iii*) non-additive discounting, when a sequence of trade-offs in a subdivided interval leads to more overall discounting than a single trade-off over the whole interval. Intermittency of waiting means that future utility is progressively but not regularly discounted with the passing of time. This slows down the discounting process and makes the consumer decreasingly impatient with the length of the delay. Present bias appears because the present is a planning date and therefore a reminding period. Contrary to future dates in which reminding is only a possibility, the agent feels the cost of waiting when the decision to postpone a consumption is made. Discounting is non-additive if reminding is certain the first date of a trade-off, when the decision to defer consumption is made, since average expected waiting costs will be higher the shorter the time interval.

Next I present a simple formula which determines a lower bound on long-term discounting. If an individual prefers consuming 1 unit of good now instead of $x > 1$ next period at every date between the present and date T, she is also ready to forego x^T units in T periods in exchange for 1 unit now. This relationship between short and long-term impatience is non parametric and applies to any discount function which decreases with delay. In several numerical examples, this lower bound leads to an excessive degree of long-term impatience as the foregone consumption x^T may potentially be quite large. This apparent paradox is solved when individuals incur intermittent waiting costs. Small departure from perfect patience over short delays need not translate into strong impatience over longer delays if distractions in daily life reduce expected waiting costs.

Last, when reminding probabilities are stationary, and waiting costs are exponentially discounted, the discount function $D(t)$ applied to date t consumption takes a simple two-parameter functional. It is equal to 1 the present date and $p\beta^t + (1-p)\beta$ for delayed consumption with p the probability of reminding the good and β a parameter measuring the propensity to resist temptation. The model boils down to the classic exponential model (Samuelson, 1937) when reminding repeats every period ($p = 1$). This provides a a behavioral foundation to the constant discounting model and makes clear the fact that present bias, decreasing impatience and sub-additive discounting are direct consequences of the assumption of intermittent waiting ($p < 1$).

The remainder of the paper is organized as follows. Section 2 reviews the related literature. Section 3 lays out a general model of consumption with intermittent waiting, poses a set of axioms and studies its consequences for time preferences. Section 4 investigates several properties of the model: present bias, decreasing impatience and non-additive discounting. Section 5 concludes.

2 Related Literature

Several models of temporal discounting have been used in the literature. The constant discounting model (Samuelson, 1937) in which date t utility is discounted by β^{-t} with $\beta \in (0, 1)$, is by far the most used in microeconomic or macroeconomic models. It is parsimonious and normatively appealing, yet has a very limited descriptive validity (see Frederick et al. (2002) and Cohen et al. (2016) for surveys).

Other models have been proposed that deviate from constant discounting. One of the most popular is quasi-hyperbolic discounting (Phelps and Pollak, 1968, Laibson, 1997) in which the discount factor is 1 the initial period and $\delta\beta^{-t}$ in all subsequent periods with $0 < \delta, \beta < 1$. Consumers are present biased, a form of decreasing impatience limited to couples of dated goods in which one is delivered now.

Loewenstein and Prelec (1992) propose a generalized model of hyperbolic discounting, in which future utility is discounted by $(1 + ht)^{-r/h}$ with $h \geq 0$ and $r > 0$. Two special cases are proportional discounting (Mazur, 1987) when $h = r$ and power discounting (Harvey, 1986) when $h = 1$.[3] In all those models, decreasing impatience not only occurs in the first period but also in later ones. Yet , hyperbolic discounting models do not give extra weight to the immediate outcome in a precise sense that will be explained in Section 4.2.

The present model contributes to this literature by investigating the consequences for time preference of a single micro-founded discounting mechanism which explains both why consumers are present biased and why they display decreasing impatience involving remote dates. As a side contribution, I propose a formal distinction between the two concepts. Although a stark example of decreasing impatience, present bias specifically involves the value of immediate gratification

[3]Bleichrodt et al. (2009) and Ebert and Prelec (2007) introduce discount functions which are the intertemporal analogues of constant absolute risk aversion and constant relative risk aversion utility and can also account for decreasing impatience.

which may have its own drivers like impulsivity, deprivation, addiction (Loewenstein and Angner, 2003) or visceral factors such as hunger, thirst, pain or sexual arousal (Loewenstein, 1996). In contrast, decreasing impatience also applies to prospects not involving immediate rewards. A formal definition and a measure of present bias are proposed, applied to the model and other models of utility discounting.

Benhabib and Bisin (2004) introduce a separate cost of delaying consumption interpreted as the psychological restraint from the impulse of choosing the immediate reward. Contrary to the wait-based model, the cost of delay is a fixed cost independent of the size of the reward and of the length of the delay. Here, the cost of waiting is proportional to deferred utility and depends on delay through the recurrence of waiting states. Laibson (2001) and Bernheim and Rangel (2004) propose models of addiction in which temptation effects endogenously depend on past associations between cues (e.g. the sight of a lighter) and rewards (smoking a cigarette). The wait-based discounting model takes as given the existence of cues, their frequency and effectiveness in triggering a behavioral response, and focuses on consequences for time preferences.

The assumption of intermittent waiting opens up the possibility of subadditive discounting, according to which a sequence of trade-offs in a subdivided interval leads to more overall discounting than a single trade-off over the whole interval. This pattern has been reported in several experiments[4] and in a German representative sample (Dohmen et al., 2012). These evidence is inconsistent with all delay-dependent discounting models, in which additivity holds regardless of the shape of the discount function.

Subadditive discounting is also shown to explain an apparent puzzle. If impatience is constant or does not decrease too fast, a plausible degree of impatience over short delays leads to an unrealistic degree of impatience over longer horizons. Intuitively, turning down the delayed option in a sequence of short delay trade-offs means that the time discounting function must diminish very quickly over longer horizons, implying a rapidly increasing degree of impatience. When discounting is intermittent, short-term impatience endogenously appears at the beginning of the trade-off when future consumption is reminded. Since the individual may be patient the rest of the time, long-term impatience may stay within reasonable bounds. The quasi-hyperbolic model has also the potential to simultaneously account for short-delay impatience and long-delay patience (Rabin, 2002, Shapiro, 2005), but only when immediate short-delay trade-offs are involved, not future ones. The present bias parameter is unable to account for some degree of impatience for short-delay trade-offs in the future, unless a very high degree of impatience over long-delay trade-offs is assumed.

The paper is also related to the vast literature in psychology on time perception. A body of consistent evidence shows that the perception of duration is affected by attention.[5] In exper-

[4]Read (2001), Read and Roelofsma (2003), Scholten and Read (2006) and Kinari et al. (2009).

[5]The father of American psychology William James noted: "The tracts of time (...) shorten in passing whenever we are so fully occupied with their content as not to note the actual time itself. (...) On the contrary, a day full of waiting, of unsatisfied desire for change, will seem a small eternity" (James, 1890). According to Stout (1932): "In general, temporal perception is bound up with the process of attention... What measures the lapse of time is

iments, the ratio of judged duration to real duration increases when attention is stimulated.[6] People who are paying attention to time itself, e.g. when they are in a queue, or when they have been told in advance to estimate a period of time, feel the time passing more slowly. On the contrary, the ratio of judged to real time decreases when subjects are kept busy by a cognitively demanding task (Zakay and Block, 1997). If attention is distracted by nontemporal information processing, less capacity is available for processing temporal information (Kahneman, 1973). Katz et al. (1991) found that distractions like watching a news board or television while waiting made the wait more acceptable for customers. Those results are connected to the article as people pay attention to time in waiting states. An increased frequency of waiting periods may be interpreted as a lengthening of the perceived temporal distance, which makes the wait subjectively longer, and in turn deepens the discount on delayed rewards.

Relatedly, some researchers argue that decreasing impatience reflects nonlinear perception of time. Ebert and Prelec (2007) report that making people pay more attention to the time dimension of the choice (e.g. by letting people focus on the arrival date of an item) has the effect of increasing discounting of the far future. Zauberman et al. (2009) find that making duration more salient to participants lead them to be more sensitive to time horizon, resulting in less similar preference between short and long time horizons.[7] In the model, intermittency of waiting makes subjective time flow non-linearly. The size of discount on future utilities does not depend on objective delay, but on the number of times the good is expected to be reminded until consumption. Compared to a model with continuous reminding, it is as if experienced time is passing only when attention to future gratifications is paid.

On the experimental side, many studies have found decreasing impatience.[8] In an experiment, Bleichrodt et al. (2016) show that a majority of subjects are decreasingly impatient and that generalized hyperbolic discounting and proportional discounting best describe time preferences. Abdellaoui et al. (2010, 2013) conclude that hyperbolic discounting perform better than constant, quasi-hyperbolic, proportional, and power discounting. Decreasing impatience has also been noted for substance abusers (Kirby, Petry, and Bickel, 1999). Based on neuroimaging, Kable and Glimcher (2007) find that hyperbolic discount functions fit behavior better than exponential discount functions.

Some experimental works have called into question the existence of present bias for monetary options (Andreoni and Sprenger (2012), Augenblick et al. (2015), Andersen et al. (2014)). Most studies retain however the same mode of payment to participants for the smaller-sooner and larger-later options, with the consequence that the sooner payment has a minimal delay. Balakrishnan et al. (2016) make the soonest available payment truly immediate and find a

the cumulative effect of the process of attending".

[6]See Fraisse, 1963, Thomas and Brown, 1974. Hicks et al. (1976) and Thomas and Weaver (1975) provide an attention-based theory of this phenomenon).

[7]See also Radu et al. (2011).

[8]Thaler (1981), Benzion et al. (1989), Green et al. (1997), Kirby (1997) or Benhabib et al. (2010)). See Attema et al. (2010) and Takeuchi (2011) for contrasting views.

substantial degree of present bias over money.

3 Intermittent discounting

3.1 Axioms

A consumer decides at which date $t \in \{0, ..., T\}$ a good is consumed. Waiting until consumption is costly. The cost occurs whenever the decision maker (DM) reminds delayed consumption. Future consumption is reminded in period $s = 0, 1, ..., T-1$ with probability $p_s \in [0, 1]$. Reminding probabilities are i.i.d. Let $(x, t) \in X \times \{0, 1, ..., T\}$ be a consumption plan devised at date 0 and consisting in consuming x at date $t \geq 0$ and zero at all other points in time. Let $\succ$ be a shorthand for strict preference relations expressed at time 0 on $X \times \{0, 1, ..., T\}$. Inverse relations $\prec$ and indifference relations $\sim$ are defined the usual way. All relations are complete transitive, and satisfy three axioms.

Axiom 1 *(Monotonicity)* $\forall x, x' \in X$ *satisfying* $x' > x$*, and* $\forall t = 0, 1, ..., T$*,* $(x', t) \succ (x, t)$*,* $\forall \{p_s; s = 0, 1, ..., T\} \in [0, 1]^{T+1}$.

Axiom 2 *(Impatience)* $\forall x \in X$ *and* $\forall t = 0, 1, ..., T$*, if* $p_t > 0$*,* $(x, t) \succ (x, t+1)$ $\forall \{p_s; s = 0, 1, ..., T, s \neq t\} \in [0, 1]^T$.

Axiom 3 *(Temporal indifference)* $\forall x \in X$ *and* $\forall t = 0, 1, ..., T$*, if* $p_t = 0$*,* $(x, t) \sim (x, t+1)$ $\forall \{p_s; s = 0, 1, ..., T, s \neq t\} \in [0, 1]^T$.

Axiom 1 ensures that the good is valuable to the DM whatever the sequence of probabilities. Axiom 2 states that if a period has a strictly positive probability of reminding, the DM always prefer consuming at this period rather than the next one. This is the classic assumption of impatience, but limited to periods in which reminding is a possibility.[9] Axiom 3 states that if a date to come is not possibly a reminding period, the DM is indifferent between consuming the good this period or the next one. It formalizes the intuition that consumption may be delayed effortlessly if individuals are distracted away. By extension, if periods t to $t + k$ have a zero probability of reminding, the DM is indifferent between consuming at dates t, $t + k + 1$, or any date within. The next sub-section proposes a model of intertemporal choice with waiting costs consistent with those axioms.

[9] It could be argued that reminding makes future gratification more salient and consequently facilitates the ability to control one-self. In their well-known experiment, Mischel and Ebbesen (1970) found to the contrary that when the rewards were out of sight, 75% of children were able to wait the full time (15 min). When it was exposed, the mean delay time was only about 1 minute. The result has been confirmed by multiple follow-up studies (Metcalf and Mischel, 1999).

3.2 Time preferences

I consider a setting in which the DM maximizes a time additive expected discounted utility function which comprises two types of utility flows: an instantaneous utility $u(x)$ (increasing and twice continuously differentiable) from consuming x, and a waiting cost $\delta(s,t)u(x)$ felt at date $s \in [0,t)$ and proportional to utility deferred to date t.[10]

Assume that consumption is delayed from date 0 to $t > 0$. Date 0 expected intertemporal utility is the sum of expected waiting costs accumulated until $t-1$ and date-t discounted utility:

$$E(\tilde{U}) = -p_0\delta(0,t)u(x) - p_1\delta(1,t)u(x) - ... - p_{t-1}\delta(t-1,t)u(x) + \gamma(t)u(x)$$

where $p_s\delta(s,t)u(x)$ is the expected utility cost felt at date s of delaying consumption until t and $\gamma(t)$ the discount applied to date t utility. For the sake of generality, expected utility is valued just before the DM observes whether the period 0 is a reminding period or not.

The formula may be generalized with an arbitrary date of consumption. Let u_t be period t utility, which may take two values: $u(x) > 0$ if the good is consumed at date t, and 0 in all other periods. Expected utility is:

$$E(\tilde{U}) = \sum_{t=0}^{T} \left(-p_0\delta(0,t) - p_1\delta(1,t) - ... - p_{t-1}\delta(t-1,t) + \gamma(t)\right) u_t \quad (1)$$

with the convention $\delta(s,t) = 0$ when $t \leq s$. How future utility is discounted depends on the whole waiting sequence through the number of times the DM reminds future consumption. The discount is stochastic since this number is uncertain.

Time preferences must satisfy Axioms 1, 2 and 3. For convenience, the consequences of Axiom 3, according to which the DM is indifferent between consuming now or later in the absence of reminding, are first derived (the proof is deferred to the Appendix):

Proposition 1 *Under Axiom 3, temporal weights in Eq. (1) satisfy:*

1. $\gamma(t+1) = \gamma(t)$, $\forall t = 0, ..., T-1$,

2. $\delta(s,t) = \delta(s,t+1)$, $\forall s = 0, ..., t-1$ *and* $t = 1, ..., T-1$.

Proposition 1.1 means equal valuation of present and future utility. To gain insight into why it is the case, consider a thought experiment. Imagine you are not thinking at the present moment of any particular good you may consume in the future. Would you be indifferent between consuming it now or the next moment? A positive answer in accordance with Axiom 3 implies equal valuation of present and future utilities.[11] A similar reasoning holds between other periods.

[10] Loewenstein (1987) assimilates waiting with the pleasure of savoring future consumption. This would imply here $\delta(s,t) < 0$, opening up the possibility of negative discounting. While relevant in particular situations, we stick here to a more common interpretation according to which people dislike waiting.

[11] Formally the indifference condition writes: $\gamma(0)u(x) = \gamma(1)u(x) - p_0\delta(0,1)u(x)$. If $p_0 = 0$, $\gamma(0) = \gamma(1)$.

Proposition 1.2 states that waiting costs depend on the date at which they will be incurred, but not on the length of the remaining delay until consumption.[12] This comes from the principle of equal valuation of utility flows but applied to disutility of waiting.

The notations may be simplified accordingly. From now on, all $\gamma(t)$, $t = 0, 1, ..., T$, are normalized to 1. With probability p_s, the DM reminds future consumption at date $s = 0, 1, ..., t-1$ and incurs the cost $\delta(s,t) = \delta_s$. Let us define the discount factor $D(t)$ as the sum of weights attached to utility enjoyed in date $t = 0, 1, 2, ..., T$ such that $E(\tilde{U}) = \sum_{t=0}^{T} D(t)u_t$. $D(0) = 1$ and

$$D(t) = 1 - p_0\delta_0 - p_1\delta_1 - p_2\delta_2 - ... - p_{t-1}\delta_{t-1} \tag{2}$$

for $t \in (0, T]$. The constraints imposed on temporal weights by the other two Axioms are derived in Proposition 2 (proof deferred to Appendix).

Proposition 2 *Under Axioms 1, 2, and 3, temporal weights* δ_s*,* $s = 0, 1, ..., T-1$*, satisfy* $1 > 1 - \delta_0 > 1 - \delta_0 - \delta_1 > ... > 1 - \delta_0 - \delta_1 - ... - \delta_{T-1} > 0$.

Axiom 1 requires that consumption is valuable at every horizon, i.e. $D(t) > 0 \; \forall t = 0, 1, ..., T$, even in the less favorable environment in which the DM waits every period before consuming, i.e. all probabilities are set to 1. Proposition 2 implies that the longer the delay until consumption, the smaller the sum of temporal weights attached to utility: $D(0) \geq D(1) \geq D(2) \geq ... \geq D(T) \geq 0$, whatever reminding probabilities $p_s \in [0, 1]$, $s = 1, ..., T-1$. The decrease of discount factors with delay is the classic definition of impatience in time discounting models. Here, the longer consumption is delayed, the less expected utility, because the greater number of periods during which the DM may remind future consumption. The decrease is however non-monotonic as she may expect periods during which future consumption is not reminded.

Propositions 2 do not say anything about the ordering of temporal weights δ_s. A non-decreasing sequence would be however counter-intuitive since future waiting should be if anything less costly than immediate ones. It would also rapidly run against the positivity constraint on $D(T)$ as the horizon extends. This point is investigated in the next subsection.

3.3 Infinite horizon

So far, the DM was assumed to have a bounded horizon. Yet, Proposition 2 should hold for an arbitrarily large number of periods, especially when the unit of time is short, like a day or an hour. New constraints on preferences appear over infinite horizon. To begin, future utility must be bounded when T grows, whatever the actual sequence of reminding probabilities. A sufficient condition is the convergence of $D(T)$ to a limit L with all probabilities set to 1:

[12] The quasi-hyperbolic model makes the case of a fixed cost on utility as the disutility of delaying an immediate reward does not vary with the horizon. See also Benhabib et al. (2010).

$\lim(1 - \delta_0 - \delta_1 - ... - \delta_{T-1}) = L$ when $T \to \infty$. This implies that temporal weights δ_s become arbitrarily close to each other as the sequence progresses.[13]

Second, the limit L to which the sequence $D(T)$ converges cannot be negative, which would violate the condition that discounted utility is positive at every horizon and for any sequence of probabilities (Proposition 2). Preferences for infinitely delayed consumption may be further constrained by the following Axiom.

Axiom 4 *(Preference with endless reminding). $\forall x, x' \in X$ satisfying $0 < u(x) < u(x') < \infty$, and $\forall t < \infty$, $(x, t) \succ \lim_{T\to\infty}(x', T)$, with all $p_s = 1$, $s = 0, 1, ...$.*

The Axiom states that when reminding repeats every period, infinitely delayed consumption is worthless, in the sense that infinitely delayed consumption is never preferred to consumption obtained within a finite delay. Apart from the restriction on reminding probabilities, this Axiom is verified by all common models of time discounting, including the exponential, hyperbolic and quasi-hyperbolic models.[14] Here the Axiom is restricted to the most adverse environment in which reminding recurs every period and expected waiting costs are maximum.[15] The Axiom implies that the limit L to which the discount function converges is 0.

Proposition 3 *Under Axioms 1, 2, 3 with $T \to \infty$, and Axiom 4, temporal weights satisfy:* $\lim_{T\to\infty}(1 - \delta_0 - \delta_1 - ... - \delta_{T-1}) = 0$.

Proof (Proposition 3) Axioms 1, 2 and 3 imply (Prop. 2): $1 > 1 - \delta_0 > 1 - \delta_0 - \delta_1 > ... > 1 - \delta_0 - \delta_1 - \delta_2 - ... - \delta_{T-1}$. With all probabilities set to 1, the early option x is preferred to the late option x' if $(1 - \delta_0 - \delta_1 - ... - \delta_{t-1})u(x) > (1 - \delta_0 - \delta_1 - ... - \delta_{T-1})u(x')$. With infinite horizon, x' is never preferred to x if $\lim_{T\to\infty}(1 - \delta_0 - \delta_1 - ... - \delta_{T-1}) = 0$ providing $u(x) > 0$ and $u(x') < \infty$. □

Proposition 3 is consistent with a declining sequence of temporal weights δ_s, revealing some form of delay discounting on waiting costs. The sequence must also decrease sufficiently fast in order not to break the positivity constraint of the discount factors $D(t)$. A natural candidate for explicitly discounting waiting costs is the exponential function.

Assumption 1 *$\delta_s = \beta^s \alpha$, $s = 0, ..., T - 1$, $\beta \in (0, 1)$, $\alpha > 0$.*

This assumption parallels the way future utility is discounted in the exponential model. Temporal weights δ_s are the product of two terms: an immediate waiting cost α per unit of utility,

[13] Using the fact that any convergent sequence is a Cauchy sequence, for any given $\varepsilon > 0$, there exists a date T_0 such that for any pair of dates (s, t) satisfying $T_0 < s < t$, we have $|D(t) - D(s)| < \varepsilon$ or $\delta_s + \delta_{s+1} + ... + \delta_{t-1} < \varepsilon$.

[14] With quasi-hyperbolic discounting, $\delta\beta^{-t}u(x) > \lim_{T\to\infty} \delta\beta^{-T}u(x')$ with $0 < \delta, \beta < 1$. Or with generalized hyperbolic preferences: $(1 + ht)^{-r/h}u(x) > \lim_{T\to\infty}(1 + hT)^{-r/h}u(x')$ with $h, r > 0$.

[15] The Axiom may not hold when reminding probabilities are only a possibility. This reflects the strict priority given to Axiom 1 according to which the utility of consumption must be positive for any sequence of probabilities. If $D(T)$ were asymptotically zero for reminding probabilities less than one, it could be negative for probabilities equal to 1.

and an exponential discount factor β^s applied to the cost incurred in s periods. Assumption 1 fits well with Propositions 1, 2 and 3. Along with Assumption 1 they guarantee the positivity of discount rates for all horizons and lead to a simple expression of the undiscounted cost of waiting.

Proposition 4 *Under Axioms 1, 2, 3 with $T \to \infty$, Axiom 4, and Assumption 1, $\alpha = 1 - \beta$.*

Proof Discount factors in Eq. (2) become with Assumption 1: $D(T) = 1 - \alpha(p_0 + p_1\beta + p_2\beta^2 + ... + p_{T-1}\beta^{T-1})$. Axiom 4 implies $\lim_{T\to\infty} D(T) = 0$ with all probabilities set to 1, or $1 - \alpha(1 + \beta + \beta^2 + ...) = 1 - \alpha/(1 - \beta) = 0$. □

The greater the discount factor β, the smaller the waiting cost per unit of utility α to preserve the convergence of the discount factor to zero. For easiness, Prop. 4 is converted into an assumption, which may hold with finite or infinite horizon:

Assumption 2 $\alpha = 1 - \beta$ *with* $\beta \in (0, 1)$.

This assumption will be occasionally used in the next section to obtain a simpler proof that wait-based preferences are decreasingly impatient and for deriving a simplified version of the model.

4 Time properties

In this section, several remarkable properties of time preferences with waiting costs are reviewed.

4.1 Decreasing impatience

Discount rates seem to decline as people consider their preferences for longer time periods. Thaler (1981) found that to delay a \$15 lottery winning for 3 months, people required an extra \$15 (277% annual discount rate); but to delay the same amount for 1 year, four times as long, they required only an extra \$45 (139% annual discount rate). In addition, as both early and late consumptions get closer to the present, people tend to assign progressively greater weight to early consumption relative to the delayed one. These preference patterns are called decreasing impatience (or hyperbolic discounting) and lead to inconsistent intertemporal choices. In contrast, constant impatience means that preference between two dated consumptions do not change if they are postponed by a common delay. Koopmans (1960) shows that it implies constant discounting, a distinctive feature of the exponential model. Most empirical results points towards decreasing impatience.

Formally, impatience is decreasing (increasing) if for any couple of dated consumption (x, t) and $(x', t + 1)$ such that the DM is indifferent between them, she prefers delaying (advancing) consumption when the two dates are shifted forward by one period.

Definition 1 (Types of impatience) *$\forall x, x' \in X$ and $\forall t \in \{0, 1, ..., T-1\}$, such that $(x, t) \sim (x', t+1)$, impatience is decreasing if $(x, t+1) \prec (x', t+2)$, constant if $(x, t+1) \sim (x', t+2)$, or increasing if $(x, t+1) \succ (x', t+2)$.*

The definition of constant impatience corresponds to the notion of stationarity introduced by Koopmans (1960) and Fishburn and Rubinstein (1982). Constant or decreasing impatience cannot be valid for any sequence of reminding probabilities. When probabilities are time varying, not only time relative to the evaluation period matters but also events occurring in calendar time. This makes difficult disentangling in decisions what comes from time preferences per se and time-varying reminding probabilities. Instead, decreasing impatience is studied in an environment in which all dates have a constant reminding probability, except the decision date in which reminding is certain:

Assumption 3 *When the DM has to choose at date 0 between (x, t) and $(x', t+1)$, $t = 0, 1, ..., T-1$ and $t' > t$, then $p_t = 1$ and $p_s = p \in [0, 1]$, $\forall s \neq t$.*

Constant reminding probabilities aim both at simplifying the setup and controlling for the effects of probabilities on time preferences. One exception regards the first date of the underlying trade-off. When the DM thinks about delaying consumption from date t to t', she may project herself at date t making her decision. This necessitates comparing utility across periods and reminding delayed consumption. The reminding probability is therefore certain the first date of a trade-off.[16] Definition 1 of decreasing impatience holds for a decision date shifted forward from date t to $t+1$. Hence $p_t = 1$ in the first trade-off and $p_{t+1} = 1$ in the second one.

Decreasing impatience will be observed if the decreasing rate of temporal weights δ_s is sufficiently high:

Proposition 5 *Under Axioms 1, 2, 3, and Assumption 3, decreasing impatience holds if*

$$\frac{\delta_t - \delta_{t+1}}{\delta_t} > \frac{p\delta_t}{D(t)} \tag{3}$$

with $D(t) = 1 - p\delta_0 - p\delta_1 - ... - p\delta_{t-1}$.

Proof According to Definition 1, decreasing impatience holds if:

$$\frac{D(t+1)}{D(t)} < \frac{D(t+2)}{D(t+1)}$$

where the left-hand side trade-off begins at date t, implying $p_t = 1$, and the right-hand side at date $t+1$, implying $p_{t+1} = 1$:

$$\frac{D(t) - \delta_t}{D(t)} < \frac{D(t) - p\delta_t - \delta_{t+1}}{D(t) - p\delta_t}$$

with $D(t) = 1 - p_0\delta_0 - p_1\delta_1 - ... - p_{t-1}\delta_{t-1}$. Prop. 5 obtains after some rearrangements. □

[16]Propositions 1, 2 and 3 are unaffected by Assumption 3, as they are valid for any sequence of probabilities.

All else equal, the smaller the reminding probabilities, the lower the right-hand side's numerator and the higher the denominator and hence the more likely decreasing impatience. Intuitively, the alternation of reminding and non-reminding periods means that the DM's attention is periodically distracted away from future rewards. This slows down the discounting process and makes her more and more patient with regard to consumption periods increasingly distant from now.

Impatience is decreasing if temporal weights are exponentially discounted (Assumption 1) and waiting costs are determined by Assumption 2:

Proposition 6 *Under Axioms 1, 2, 3, Assumptions 1, 2 and 3, impatience is decreasing (constant) if* $p \in [0, 1)$ *(*$p = 1$*).*

Proof If $\delta_s = \alpha\beta^s$ (Assumption 1), and $\alpha = 1 - \beta$ (Assumption 2), Condition 3 in Prop. 5 becomes:

$$1 - \beta > \frac{p(1-\beta)\beta^t}{1 - p(1-\beta)(1 + \beta + \beta^2 + ... - \beta^{t-1})}$$

implying $1 - p(1 - \beta^t) > p\beta^t$, and consequently $1 - p > 0$. □

Temporal weights decrease relatively rapidly with exponentially discounted waiting costs. This was a condition for the positivity of the discount function when the horizon grows larger. The same property is also necessary for decreasing impatience. Interestingly, impatience is constant in the limit case in which reminding repeats every period ($p = 1$). In this sense, decreasing impatience rests on the assumption of intermittent reminding.

4.2 Present bias

Individuals discount the value of delayed consumption more heavily when delaying an immediate consumption (e.g., from today to next week) than when delaying the same consumption over an equal delay starting at a later date (e.g., from 30 days from today to 30 days plus one week from today). This may be interpreted as an 'immediacy effect', i.e. subjects value rewards significantly higher when they are obtained immediately. The distinction is supported by neuroimagining studies. McClure et al. (2007) show that limbic reward-related areas of the brain have greater activity when an intertemporal choice includes an immediate reward than when the options include only delayed rewards.

The strong appeal to consuming now vs in any future dates (next hour, week or month) is a stark instance of decreasing impatience. Yet present bias specifically involves the value of immediate gratification which may have its own drivers like impulsivity, deprivation, addiction, or transient visceral factors such as hunger, thirst, pain or sexual arousal. On the contrary, decreasing impatience also applies to prospects not involving immediate rewards, as attested by Definition 1.

This motivates a specific analysis of present bias. A formal definition starts from the assumption that the model's first period has a variable duration denoted μ, which may be arbitrarily narrowed, while other periods have an unchanged length.

Definition 2 (Present bias) *Preferences are present biased if* $\forall x \in X$, $(x, 0) \succ \lim_{\mu \to 0^+}(x, \mu)$.

A present biased DM does not value identically an immediate payoff and a payoff delivered next period, where next period can be as soon as desired. This is an extreme form of decreasing impatience which embodies the strong appeal of immediate consumption. It leads to a simple measure of present bias.

Definition 3 (Measure of present bias) *Let* $D(t)$ *be a discount function. A measure of present bias is* $PB = \lim_{\mu \to 0^+} \dfrac{D(0)}{D(\mu)} - 1$.

PB is positive in case of present bias and negative in case of future bias. It is higher the larger the discrepancy between the time valuation of immediate vs temporally close utility. In the exponential model where $D(t) = (1+\rho)^{-t}$ and ρ is the subjective discount rate defined over a unit period of time, the measure is zero:

$$PB = \lim_{n \to \infty} \frac{1}{\dfrac{1}{1+\frac{\rho}{n}}} - 1 = 0$$

with n the number of sub-intervals into which the unit interval is partitioned. Likewise, hyperbolic preferences of Loewenstein and Prelec (1992) do not display a jump discontinuity at $t = 0$ (with $h \geq 0$ and $r > 0$):

$$PB = \lim_{\mu \to 0^+} \frac{(1+0)^{-r/h}}{(1+h\mu)^{-r/h}} - 1 = 0$$

Although a prominent example of decreasing impatience, it is not present biased according to Definition 2. To the contrary, all models which have a discontinuity at date 0, like the quasi-hyperbolic model of Laibson (1997), its continuous version in Harris and Laibson (2013), or the fixed cost model of Benhabib et al. (2010), display a positive measure. As an illustration, the measure for quasi-hyperbolic preferences is (with the notations $D(0) = 1$ and $D(t) = \delta(1+\rho)^{-t}$, for $t > 0$ with $0 < \delta < 1$ and $\rho > 0$):

$$PB = \lim_{n \to \infty} \frac{1}{\dfrac{\delta}{1+\frac{\rho}{n}}} - 1 = \frac{1}{\delta} - 1 > 0$$

The larger the present bias parameter $1/\delta$, the larger the measure PB. In the wait-based model, the decision date is assimilated as a reminding date (Assumption 3). Here this means that reminding is certain in the present ($p_0 = 1$). Preferences are also present biased in this case.

Proposition 7 *Under Axioms 1, 2 and 3, and Assumption 3, preferences are present biased.*

Proof According to Definition 2, preferences are present biased if $D(0)u(x) > \lim_{\mu \to 0^+} D(\mu)u(x)$ or with $p_0 = 1$: $1 - (1 - \delta_0) > 0$, or $\delta_0 > 0$. □

The date 1 temporal weight is $D(1) = 1 - \delta_0$ which is insensitive to any narrowing of the duration between date 0 and date 1. This is a direct consequence of Proposition 1.2 which states that the waiting costs depend on the date at which they are felt, but not on the length of the further delay until consumption. The DM feels the need now, irrespective of when the good will actually be consumed, in one month or one hour. The present bias intensity as measured by Definition 3 is:

$$PB = \frac{1}{1 - \delta_0} - 1$$

The measure is positive in presence of waiting costs ($\delta_0 > 0$). The larger the waiting costs, the higher the present bias.

As a consistency check, one may wonder to what extent the bias is specific to the initial date. Let us suppose that instead of folding the time-line at date 0, it is done at date $t > 0$. Would a discontinuity between $D(t)$ and $D(t + \mu)$ be observed? The answer is negative, as long as reminding remains uncertain. As the period length between date t and $t + \mu$ shrinks, so does the probability of feeling the wait:

$$\lim_{\mu \to 0^+} p_{t-1+\mu} = 0 \tag{4}$$

except if $p_{t-1+\mu} = 1$. Hence:

$$\lim_{\mu \to 0^+} \frac{D(t)}{D(t+\mu)} - 1 = \lim_{\mu \to 0^+} \frac{1 - p_0\delta_0 - p_1\delta_1 - ... - p_{t-1}\delta_{t-1}}{1 - p_0\delta_0 - p_1\delta_1 - ... - p_{t-1}\delta_{t-1} - p_{t-1+\mu}\delta_{t-1+\mu}} - 1 = 0$$

The immediacy effect is therefore specific to the initial date under Assumption 3 and with trade-offs involving the present date. It also follows that a similar present bias would occur in the future whenever the DM expects reminding consumption with certainty since the limit probability in Eq. 4 does not converge to zero. Furthermore, in the limit case in which reminding repeats every period with certainty, all future dates are present biased. It is therefore equivalent to a model without present bias as no period stand out from others.[17]

4.3 Preferences over short and long delays

A common feature of all models of time discounting is a strong relation between short and long-term impatience, which may be questionable in several instances. Let x be the maximal quantity the DM, questioned at date 0, is willing to forego at date $t + 1$ in exchange for a unit

[17] Consistent with this interpretation, a simplified version of the model in which reminding repeats every period is shown in the next sub-section to be equivalent to the exponential model. A similar property could also be observed in other present biased models. In the spirit of Laibson (1997), assume that the DM is aware that a present bias will appear in all future dates and that preferences are consistent with this view. The discount factor is $(1/\delta)(1 + \rho)^{-t}$ in all periods, including the present one ($t = 0$), with $1/\delta$ the extra weight given to the present period and all future 'present periods'. It is then equivalent to the exponential model with no present bias and a discount factor equal to $(1 + \rho)^{-t}$.

quantity at date t where t is any date between now and $T-1$. Assuming intertemporal utility is time separable, period utility u is linear, and noting $D(t)$ the discount factor attached to date t utility, x is such that for any $t \in [0, T-1]$, $D(t+1)x \leq D(t)$ with at least one equality, or:

$$\frac{D(t)}{D(t+1)} \geq x \qquad t = 0, 1, ..., T-1$$

with $x \geq 1$ if the DM is impatient. It is possible to infer from these inequalities a lower bound for the maximal quantity y the DM is willing to forego at date T in exchange for a unit quantity at date 0:

$$\frac{D(0)}{D(T)} = \frac{D(0)}{D(1)}\frac{D(1)}{D(2)}...\frac{D(T-1)}{D(T)} = y \geq x^T \tag{5}$$

Long-term discount rates may be decomposed as a sequence of short-term discount rates the same way long-term interest rates are the product of short-term interest rates in financial markets. The equality is non-parametric and applies to any discount function representing time additive preferences, including hyperbolic or quasi-hyperbolic models.

In plausible situations, the quantity x^T may be judged too large. For instance, if at any time over a week the DM prefers eating one chocolate bar now rather than $x = 1.02$ chocolate bar one hour later, a seemingly reasonable amount, she is also willing to forego $1.02^{7\times 24} = 28$ chocolate bars in one week in exchange for a single bar now.

Although, there seems to be no experiment specifically designed to test whether short-delay discount rates estimated at different points in time are consistent with long-delay discount rates, some evidence suggests a quantitative puzzle. Halevy (2015) documents a stable discounting rate of approximately 4% over a one-week interval for an immediate trade-off and a second one deferred four weeks later. Assuming that time preferences are still stationary for trade-offs delayed up to one year, this would imply $y \geq 1.04^{54} = 8.3$, a very large amount. Kinari et al. (2009) estimate a discount rates over a two-week interval for various delays at least equal to 6.5% leading to $y \geq 1.065^{22} = 4$ which is still very large. In both studies, an uncertainty exists however whether estimated discount rates are stable for longer delays.

Rubinstein (2003) is closest in spirit to the current analysis. In an experiment conducted in the Fall of 2002, subjects had to choose between two options: (a) receiving $467 on June 17th 2004 or (b) $607.07 on June 17th 2005. Other subjects had to choose between: (a) receiving $467 on June 16th 2005 or (b) $467.39 on June 17th 2005. A majority of subjects chose the early option in the second question. The difference between $467 and $467.39 is small enough that turning down the greater sum even for one day seems justified. Now suppose that the subjects would also take the early option if the trade-off had taken place *anytime* between June 17th 2004 and June 17th 2005.[18] With linear utility and for any $t \in [t_0, t_{365}]$:

$$\frac{D(t)}{D(t+1)} \geq \frac{467.39}{467} = 1,000835$$

[18] This assumption departs from the experiment but is realistic, especially if subjects are decreasingly impatient.

which reveals a 0.083% daily discount rate. A lower bound for the long-delay trade-off is computed using Eq. 5:

$$\frac{D(t_0)}{D(t_{365})} \geq 1,000835^{365} = 1,35620$$

which translates into an annual discount rate of at least 35.6%. All subjects choosing the early outcome in the one-day trade-off should also choose the early outcome in the long-delay trade-off of the first question, since the latter implies a smaller discount factor:

$$\frac{D(t_0)}{D(t_{365})} \geq \frac{607.07}{467} = 1,29994$$

Yet, a majority of respondents chose the late outcome. Rubinstein shows that choosing delay in the first question and no delay in the second question is inconsistent even according to the more general hyperbolic discounting utility function. A similar argument is made here without posing any functional forms for the discounting function.

In all those examples, a plausible degree of impatience over short delays leads to an unrealistic degree of impatience over longer horizons. Intuitively, turning down the delayed option in short delay trade-off means that the time discounting function must diminish very quickly over longer horizons, implying a rapidly increasing degree of impatience.

Assuming concave utility does not close the gap between x and y. Posing $u(1) = 1$, x is now defined by the set of inequalities $D(t+1) \leq u(x)D(t)$. The minimal quantity y the DM is willing to consume at date 0 instead of a unit quantity at date T is bounded by:

$$\frac{D(0)}{D(T)} = u(y) \geq u(x)^T$$

Let us assume the functional form $u(x) = x^\sigma$ with $\sigma \in (0,1)$. We are back to the previous inequality whatever the degree of concavity: $x \leq y$. Other functional forms are unlikely to make a significant difference.

I see two possible answers to what looks like an anomaly. The first one is to admit that the DM must be perfectly patient or very close to, when confronted with short delay trade-offs delayed in the future. This is defensible on normative grounds but still an open issue on the empirical side.

Alternatively, a long-delay trade-off could be treated differently by the DM than a sequence of short-delay trade-offs. The latter involves $T-1$ periods of sure waiting whereas the former only imposes an initial waiting period followed by a long period where waiting alternates with other activities. The first option should be more painful for an impatient DM than the second one. Let us illustrate the mechanism in a two period model. Eq. 5 with $T = 2$ rests on an equivalence between (i) discounting the good from date 2 to date 1, then further discounting it from date 1 to the present, and (ii), directly discounting it from date 2 to the present. This equivalence breaks if discounting is subadditive (Read, 2001), i.e. if the sequence of one-period trade-offs leads to more overall discounting than a single trade-off over two periods. Let us denote $D_{[t_0,t_1]}(t)$ the discount applied to date $t \in \{t_0, t_1\}$ utility for a trade-off between t_0 and

t_1. Non-additivity arises if:

$$\frac{D_{[0,2]}(2)}{D_{[0,2]}(0)} \neq \frac{D_{[1,2]}(2)}{D_{[1,2]}(1)}\frac{D_{[0,1]}(1)}{D_{[0,1]}(0)} \tag{6}$$

With intermittent waiting costs, discounting is non-additive if the expected sequence of reminding varies with the timing of the trade-off, as it is the case with Assumption 3. If the DM expects reminding the good with certainty the first date of the trade-off, Eq. 6 becomes:

$$1 - \delta_0 - p\delta_1 \neq \frac{1 - p\delta_0 - \delta_1}{1 - p\delta_0}\frac{1 - \delta_0}{1} \tag{7}$$

The left-hand side term is greater than the right-hand side, i.e. discounting is subadditive, if waiting costs and/or reminding probabilities in non-decision dates are small enough.[19] Intuitively, total waiting costs are inferior over a two-period trade-off than two one-period trade-offs if the DM expects to be distracted at date 1 with a sufficient probability or if waiting costs in the long-delay trade-off are not too high.

Subadditivity means that a lower bound for the amount y in Eq. 5 can no longer be retrieved from inequalities over short-delay trade-offs, because it is based on the false premise that the expected sequence of waiting is the same for the short and long delay trade-offs.

With more than two periods, discounting is also subadditive if reminding probabilities in non-decision dates are not too high. For simplicity, let us set these probabilities to zero, i.e. $p_s = 0 \; \forall s \neq t$ for a trade-off between t and t'. This is Assumption 3 with $p = 0$ at every date except when the decision is made. Preferring 1 at date $t \in [0, T-1]$ instead of x next period means:

$$\frac{D_{[t,t+1]}(t)}{D_{[t,t+1]}(t+1)} = \frac{1}{1-\delta_t} \geq x$$

If temporal weights are decreasing ($\delta_t > \delta_{t+1}$), the maximal quantity of good the DM is willing to forego next period is determined by the last-date trade-off, at a time when she is the most patient:

$$x = \frac{1}{1-\delta_{T-1}}$$

The amount y is determined by:

$$y = \frac{D_{[0,T]}(0)}{D_{[0,T]}(T)} = \frac{1}{1-\delta_0}$$

y is larger than x when temporal weights are decreasing ($\delta_0 > \delta_{T-1}$), which is intuitive, as waiting over one period is less costly than over T periods, especially when the one-period delay takes place far in the future. Yet y may not need to rapidly grow with horizon, even when the DM shows some impatience at short-term frequency.

As an illustration, suppose temporal weights satisfy Assumption 1: $\delta_t = \beta^t\alpha$, with $\alpha > 0$ the undiscounted waiting cost and $\beta \in (0,1)$ the one-period discount factor applied to future waiting costs: $x = (1-\alpha\beta^{T-1})^{-1}$ and $y = (1-\alpha)^{-1}$. The first numerical example with $x = 1.02$, $T = 168$ and, say, $y = 2$ leads to plausible preference parameters: $\alpha = 1/2$ and $\beta = 0.98$.

[19] The inequality 7 simplifies to $1-\delta_0-p(1-p\delta_0) > 0$. Let us define $f(p) = 1-\delta_0-p(1-p\delta_0)$. $f(0) = 1-\delta_0 > 0$, $f(1) = 0$ and $f'(p) = -1 + 2p\delta_0 < 0$ is verified if p and/or δ_0 are small enough.

4.4 A two-parameter discount function

In this subsection, various assumptions studied separately are considered together, with the aim of making the model defined by Propositions 1 and 2 more operational and tractable. Temporal weights are interpreted as discounted waiting costs (Assumption 1). The discount factor asymptotically converges to zero with infinite horizon (Assumption 2). Reminding probabilities are constant, except at the beginning of trade-offs when reminding is certain (Assumption 3). All decisions are made in the present, hence $p_0 = 1$.

Proposition 8 *Under Assumptions 1, 2 and 3, discount factors are:* $D(0) = 1$ *and* $D(t) = (1-p)\beta + p\beta^t$, $\forall t > 0$.

Proof Discount factors in Eq. 2 become with $\delta_s = (1-\beta)\beta^s$, $s = 0, ..., t-1$, (Assumptions 1 and 2), $p_0 = 1$ and $p_t = p \in [0, 1]$ (Assumption 3), $D(t) = 1-(1-\beta)-p(1-\beta)(\beta+\beta^2+...+\beta^{t-1}) = (1-p)\beta + p\beta^t$. □

Intertemporal expected utility is:

$$E(\tilde{U}) = u_0 + \sum_{t=1}^{\infty} \left(p\beta^t + (1-p)\beta\right) u_t$$

The parameters β and p relate to two distinct dimensions of time preferences: the discounted cost of waiting (or the 'amount' of discounting) and the frequency with which future rewards are reminded (or the 'speed' of discounting). The extreme case in which $\beta = 0$ corresponds to perfect shortsightedness as only current utility matters: $E(\tilde{U}) = u_0$.[20] At the other extreme, $\beta = 1$ implies zero waiting costs and corresponds to a perfectly patient DM whose intertemporal utility is the undiscounted sum of present and future period utilities: $E(\tilde{U}) = \sum_{t=0}^{\infty} u_t$.

The probability p controls the frequency with which future utility is progressively discounted until the consumption date. In the limit case in which $p = 0$, preferences are present biased: $E(\tilde{U}) = u_0 + \beta \sum_{t=1}^{\infty} u_t$. The DM has a strict preference for the present but is perfectly patient when future prospects are compared. She incurs an immediate waiting cost which depreciates future utility by the factor β, after deduction of waiting costs $(1-\beta)u(x)$. This is a special case of the quasi-hyperbolic model (Laibson, 1997) in which the absence of future episodes of reminding leaves future utilities undiscounted.

At the opposite, if future consumption is reminded every period ($p = 1$), future utility is exponentially discounted: $E(\tilde{U}) = \sum_{t=0}^{T} \beta^t u_t$. The exponential model is an interesting special case as it corresponds to a cornerstone of microeconomic and macroeconomic models of intertemporal choices. It receives a behavioral interpretation based on recurring waiting costs. The DM shows a strong impatience in the sense that future consumption is reminded every period. Overall, the

[20] Here waiting costs are maximum and equal to delayed utility of the good. At date 0, the DM sets a consumption plan, reminds future utility and incurs waiting costs equal to future utility. All subsequent waiting costs are zero due to the application of a zero discount.

four polar cases in which the two parameters β and p take extreme values correspond to very different time preferences. This demonstrates the existence of two distinct dimensions of time discounting.

The wait-based discount function $D(t)$ may also be read literally as the expectation of a random factor equal to β^t with probability p and $\beta < 1$ with probability $1 - p$. The consumer is endowed with standard time additive intertemporal preferences except she is uncertain about the appropriate discount function to apply in future dates. In reminding periods, utility is exponentially discounted whereas in non-reminding dates, utility is less discounted and only reduced by the constant $\beta \geq \beta^t$. The duality may reflect the conflict of two selves or systems. One self is the deliberative and controlled decision-making system and discounts exponentially. The second is present biased and time inconsistent, although more patient. McClure et al. (2007) offer a similar interpretation in the quasi-hyperbolic model. Other models of two selves are studied by Metcalfe and Mischel (1999) and Ainslie (1992).

5 Conclusion

The paper proposes a novel theory of time discounting in which intertemporal trade-offs depend on two dimensions: how much future utility is discounted when future consumption is reminded and how many times the good is expected to be reminded until the consumption date. Reminding is costly and acts as an additional layer of discounting on future utility. The more consumption is delayed, the less expected utility, because the greater number of periods during which the DM may remind future consumption.

The model explains why children in the marshmallow experiment waited much longer for a preferred reward when they were distracted from it than when they attended to it (Mischel et al., 1972), or why repeated exposure to temptation goods in advertising campaigns often lead consumers to give in. It also accounts for recent results by Hofmann et al. (2012) and Ent et al. (2015) that people who were the best at self-control reported fewer temptations rather than better ability to resist temptations.

In addition, a single wait-based psychological mechanism accounts for two important features of time preferences: present bias and decreasing impatience. A separate definition and measure of present bias is also proposed, which stresses its specificity compared to the broader definition of decreasing impatience. The model also proposes a behavioral interpretation of the classic model of constant discounting in which consumers remind future gratifications every periods.

While leading economic models of discounting implicitly treats time as a continuous flow, the wait-based model adopts a nonlinear approach, more familiar to psychologists, in which experienced time elapses only when attention to future gratifications is paid. It provides a natural explanation of why discounting is subadditive, with important consequences for the link between short and long delay discounting. Pure discount rates for long horizons may be disconnected from discount rates over short delays if the later involves proportionately more waiting than the former. The convenient parallel often made between the psychological rate and the market rate used in financial planning falls short when discounting is intermittent. Familiar methods like continuous compounding or annualization of subjective discount rates estimated over different horizons may prove dubious in this case.

Researchers have documented sharp differences in intertemporal choice across individuals, groups and tasks (Fredericks et al., 2002). These are primarily interpreted as variability in discount rates although a significant source of heterogeneity might come from the frequency with which individuals remind future gratifications. For instance, smokers (Baker et al., 2003), alcoholics (Vuchinich et al., 1998) or substance-dependent individuals (Kirby et al., 1999) show large discounting of delayed rewards but also remind addictive good many times a day. The paper does not explain why some people experience fewer temptation episodes than others. Addiction, weak habits, or genetic predispositions are part of the explanation. Some people seem seem also better able to avoid potential conflicts, e.g. by installing adaptive routines (Gillebaart and de Ridder, 2015). It would be interesting to design experiments which would separately estimate these two dimensions of time preferences. Another avenue for future research is to investigate the implications of wait-based preferences in consumption-saving models.

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Proofs of Propositions

Proof of Proposition 1

Proposition 1: *Under Axiom 3, temporal weights in Eq. (1) satisfy: (i)* $\gamma(t+1) = \gamma(t)$ $\forall t = 0, ..., T-1$, *and (ii)* $\delta(s,t) = \delta(s,t+1)$, $\forall s = 0, ..., t-1$ *and* $t = 1, ..., T-1$.

Proof: Consuming x at date 0 or 1 are equivalent if $\gamma(0)u(x) = -p_0\delta(0,1)u(x) + \gamma(1)u(x)$. With $p_0 = 0$, $\gamma(0) = \gamma(1)$. Consuming x at date 1 or 2 are equivalent if

$$-p_0\delta(0,1)u(x) + \gamma(1)u(x) = -p_0\delta(0,2)u(x) - p_1\delta(1,2)u(x) + \gamma(2)u(x)$$

With $p_1 = 0$, $\gamma(2) - \gamma(1) - p_0[\delta(0,2) - \delta(0,1)] = 0$. The equality must hold whatever $p_0 \in [0,1]$, hence $\gamma(2) = \gamma(1)$ and $\delta(0,2) = \delta(0,1)$. Likewise, indifference between dates $t = 2, ..., T-1$ and $t+1$, with $p_t = 0$, implies

$$\gamma(t+1) - \gamma(t) - p_0[\delta(0,t+1) - \delta(0,t)] - p_1[\delta(1,t+1) - \delta(1,t)]$$

$$- ... - p_{t-1}[\delta(t-1,t+1) - \delta(t-1,t)] = 0$$

These equalities must hold whatever reminding probabilities $\{p_0, p_1, ..., p_{t-1}\} \in [0,1]^T$. Hence for any $s = 0, ..., t-1$ and $t = 1, ..., T-1$: $\delta(s,t) = \delta(s,t+1)$, and for any $t = 1, ..., T-1$: $\gamma(t) = \gamma(t+1)$. □

Proof of Proposition 2

Proposition 2: *Under Axioms 1, 2, and 3, temporal weights in Eq. (2) satisfy* $1 > 1 - \delta_0 > 1 - \delta_0 - \delta_1 > ... > 1 - \delta_0 - \delta_1 - \delta_2 - ... - \delta_{T-1} > 0$.

Proof: Under Axiom 1, x' is strictly preferred to x at date $t = 0, 1, ..., T$ if

$$(1 - p_0\delta_0 - ... - p_{t-1}\delta_{t-1})(u(x') - u(x)) > 0$$

This holds for any vector of probabilities $(p_0, p_1, ..., p_{t-1}) \in [0,1]^t$ if

$$1 - \delta_0 - \delta_1 - ... - \delta_{t-1} > 0 \qquad t = 0, 1, ..., T. \tag{8}$$

Under Axiom 2, consuming x at date 0 is strictly preferred to x at date 1 if $u(x) > (1-p_0\delta_0)u(x)$, or $p_0\delta_0 > 0$. $p_0 > 0$ implies $\delta_0 > 0$. Similarly, consuming x at date $t = 1, ..., T-1$ is strictly preferred to x at date $t+1$ if $(1 - p_0\delta_0 - ... - p_{t-1}\delta_{t-1})u(x) > (1 - p_0\delta_0 - ... - p_t\delta_t)u(x)$, or $p_t\delta_t > 0$. $p_t > 0$ implies $\delta_t > 0$. Together with inequality 8, they prove Prop. 2. □