Countercyclical versus Procyclical Taylor Principles

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Abstract

Assuming inflation is a forward variable in Taylor (1999) model, this paper finds opposite policy rule recommandations with counter-cyclical policy rule parameters (Taylor principle: inflation rule larger than one and bounded upwards) in the case of optimal policy under commitment versus pro-cyclical policy rule parameters (inflation rule parameter below zero) in the case of discretionary policy. For the observed high inertia of the Fed with tiny variations of the nominal policy rate within the range [0%,4%] during the great moderation, the cost of time-inconsistency is negligible for optimal policy without commitment. In this case, time-inconsistency cannot be the ultimate argument to reject counter-cyclical Taylor principle.

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1 Introduction

What is the best monetary policy? This paper revisits the classic comparison between Ramsey optimal policy under commitment (Simaan and Cruz (1973), Kydland and Prescott (1980), Miller and Salmon (1985), Miller

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(1985), Backus and Driffill (1986)) and discretionary policy (Cohen and Michel (1988), Oudiz and Sachs (1985), Backus and Driffill (1986)), see a recent comparison in Leith and Wren-Lewis (2013) among others. The new tool is to compare observationally equivalent reduced form Taylor rules for each of the two policies. The focus is the range of values of the inflation Taylor rule parameter obtained by each of these two policies.

Taylor's (1999) model is the simplest possible dynamic model of monetary policy with an interest rule describing the Federal Reserve's account of its actions. The Fed's policy interest rate has a negative effect on current output gap. Current output gap increases next period expected inflation. Cochrane (2011 and 2016) refers to Taylor's (1999) as the benchmark model of old Keynesian countercyclical monetary policy. Inflation and output are predetermined variables, but Taylor (1999) mentions that "*these equations summarize more complex forward-looking models*" (p.662). We turn Taylor's (1999) model into a forward-looking model, where expected inflation is a forward variable. In the early eighties, inflation in the USA fell like a stone during Volcker's mandate: it is difficult to explain the magnitude of this fall by old-Keynesian adaptive expectations where expectations depends only on past values of inflation. This allows to find closed forms solutions, two dimensions phase diagrams and the feasible sets of inflation rule parameter (Taylor principles) for optimal policy under commitment, discretionary policy and simple rules.

We find opposite Taylor principles with optimal policy under commitment (with a countercyclical inflation rule parameter) with respect to discretionary policy (with a procyclical inflation rule parameter). This result *matters a lot* for policy maker's advice. *Procyclical* interest rule parameters are the opposite of the explicitly *countercyclical* stabilizing language in the Federal Reserve's account of its actions. In simple rule equilibrium with expected inflation as a forward variable, "*higher inflation leads the Fed to set interest rates in a way that produces even higher future inflation. For only one value of inflation today will inflation fail to explode, or, more generally leave a local region. Ruling out nonlocal equilibria, new-Keynesian modelers conclude that inflation today must jump to the unique value that leads to a locally bounded equilibrium path*" (Cochrane (2011), p.566). This paper adds discretionary policy as another theory of *procyclical positive feedback* inflation Taylor rule parameter, besides simple rule equilibrium with expected inflation as a forward variable.

Cochrane (2011) mentions two theories for *countercyclical negative feedback* inflation Taylor rule parameter: the fiscal theory of the price level (Leeper (1991)) and the old Keynesian theory which assume that expected inflation and prices are predetermined variables instead of being forward variables (Taylor (1999)). This paper adds a third theory: Ramsey optimal policy under commitment. With this theory, a *mathematical* result is that the inflation rule parameter is *identical* to the one found by optimal policy *as if* all private sector's forward variables were assumed to be predetermined.

The observed high inertia of the Fed with tiny variations of the nominal policy rate within the range [0%,4%] during the great moderation may suggest Fed's preference for a high relative cost of changing the policy rate during this period. In this case, we show that the cost of time-inconsistency and, hence, the requirement of Fed's commitment are negligible for optimal policy. In this case, time-consistency is *no longer* the ultimate criterion for advising discretionary policy with procyclical inflation rule parameter instead of optimal policy under commitment with countercyclical inflation rule parameter.

Section 2 presents equilibria hypothesis using Taylor's model. Section 3 to 7 finds solutions of these equililbrium. Section 8 compares optimal policy under commitment and discretionary policy with respect to Taylor principles, welfare, zero lower bound and time-inconsistency issues.

2 Policymaker Equilibria with Taylor's (1999) model

Taylor (1999) model assumes that current output gap x_t depends negatively (parameter $-\delta$) on the current nominal policy rate i_t minus current inflation π_t and an additive identically and independently distributed normal component $\varepsilon_{x,t}$ of a random productivity shock $z_{x,t}$:

$$x_t = -\delta\left(i_t - \pi_t\right) + z_{x,t} \text{ where } \delta > 0$$

The productivity shock $z_{x,t}$, also called forcing variable, includes an autoregressive component (autocorrelation ρ):

$$z_{x,t} = \rho z_{x,t-1} + \varepsilon_{x,t} \text{ where } 0 < \rho < 1 \text{ and } \varepsilon_{x,t} \text{ i.i.d. normal } N\left(0, \sigma_x^2\right)$$

Future inflation π_{t+1} increases with current inflation and current output gap plus a random cost push shock $\varepsilon_{\pi,t}$, with possibly a non-zero covariance with the productivity shock:

$$\pi_{t+1} = \pi_t + \kappa x_t + \varepsilon_{\pi,t} \text{ where } \kappa > 0 \text{ and } \varepsilon_{\pi,t} \text{ i.i.d. normal } N\left(0, \sigma_\pi^2\right)$$

Eliminating output gap reduces the model to a two-equations system:

$$\pi_{t+1} = (1+\sigma)\,\pi_t - \sigma i_t + z_t \text{ where } 0 < \sigma = -\kappa\delta \tag{1}$$

$$z_t = \rho z_{t-1} + \varepsilon_t \text{ where } 0 < \rho < 1 \text{ and } \varepsilon_t \text{ i.i.d. normal } N\left(0, \sigma^2\left(\varepsilon_{x,t} + \varepsilon_{\pi,t}\right)\right) \tag{2}$$

We change Taylor (1999) model assuming inflation is a forward variable with unknown initial condition in order to model rational expectations, jumps of expected inflation and the related issue of time-inconsistency of the policymaker. The forcing variable z_t is assumed to be predetermined with given initial value z_0 and bounded with an autoregressive parameter strictly between zero and one (initial and final boundary conditions:

$$z_0 \text{ predetermined} \tag{3}$$

$$\lim_{t\to+\infty} \beta^t E_t\left(z_t\right) = 0. \tag{4}$$

The Fed determines the policy rate i_t and the path of inflation π_t minimizing a discounted quadratic loss function L_0^* with a discount factor β and relative weight $Q_{\pi z} \geq 0$ on the covariance of inflation with the forcing variable and a strictly positive relative cost of changing the policy rate $R > 0$ subject to the private sector's model (with an optimal value function $v(\pi_0, z_0)$):

$$v(\pi_0, z_0) = \max_{\{i_t, \pi_t\}} -\frac{1}{2} E_t \sum_{t=0}^{+\infty} \beta^t \left[\pi_t^2 + 2Q_{\pi z}\pi_t z_{\pi,t} + R i_t^2\right] \tag{5}$$

We compare dynamics, policy rules and Taylor principles of these equilibria ordered by their degree of optimality:

(1) Optimal policy under commitment (Simaan and Cruz (1973), Kydland and Prescott (1980)): The Fed minimizes the loss function (5) subject to private sector's law of motion (1, 2) and boundary conditions (3,4) and to two initial and final transversality conditions for the forward variable inflation (6, 7). The Fed chooses the optimal initial and final value of inflation minimizing the optimal value of her loss function at the initial and the final date. The marginal value of Fed's optimal loss function (equal to the Lagrange multiplier of inflation $\mu_{\pi,t=0}$) is set to zero at the initial date and for its infinite horizon limit. Hence, the Lagrange multiplier of inflation is a second predetermined variable besides the forcing variable z_t. This implies that expected inflation is bounded:

$$\frac{\partial v(\pi_0, z_0)}{\partial \pi_0} = 0 = \mu_0 \text{ predetermined} \Leftrightarrow \pi_0 = \pi_0^* \quad (6)$$

$$\lim_{t \to +\infty} \frac{\partial v(\pi_t, z_t)}{\partial \pi_t} = 0 = \lim_{t \to +\infty} \beta^t \mu_t \Leftrightarrow \lim_{t \to +\infty} \pi_t = \lim_{t \to +\infty} \pi_t^* \quad (7)$$

For example, Bryson and Ho ((1975), p.55) explain these transversality conditions. "*If* π_t *is not prescribed at* $t = 0$, *it does not follow that* $\delta\pi_0 = 0$. *In fact, there will be an optimum value for* π_0 *and it will be such that* $\delta v = 0$ *for arbitrary small variations of* π_0 *around this value. For this to be the case, we choose* $\frac{\partial v}{\partial \pi_0} = \mu_0 = 0$ *(1) which simply says that small changes of the optimal initial value of the forward variables* π_0 *on the loss function is zero. We have simply traded one boundary condition:* π_0 *given, for another, (1). Boundary conditions such as (1) are sometimes called "natural boundary conditions" or transversality conditions associated with the extremum problem."* When using the Lagrange multiplier solution, the policy maker's Lagrange multipliers of private sector's forward inflation is *predetermined at the value zero:* $\mu_0 = 0$. Hence, the policy maker's Hamiltonian system includes a number of predetermined variables which is equal to the number of private sector's predetermined variables and the number of policymaker's Lagrange multiplier of each of the private sector's forward variables. The number of policy maker's predetermined variables is equal to the sum of the number of private sector's predetermined *and* forward variables.

(2) Discretionary policy (Oudiz and Sachs (1985): The Fed minimizes the loss function (5) subject to private sector's law of motion (1, 2) and boundary conditions of the forcing variable (3, 4) and subject to two additional constraints: a private sector inflation rule (8) and a policy rule (9). First, it is assumed that the private sector inflation rule is a linear function of the predetermined variable with an optimal bounded parameter N_D to find:

$$\pi_t = N_D z_t \quad (8)$$

Equation (8) implies that transversality conditions on expected inflation (6 and 7) are no longer useful. Second, the policy maker's interest rule is a linear function of the predetermined forcing variable with an optimal bounded forcing variable Taylor rule parameter $F_{z,D}$ to find, or, alternatively, an optimal bounded inflation Taylor rule parameter $F_{\pi,D}$ to find:

$$i_t = F_{z,D} z_t = F_{\pi,D} \pi_t \text{ with } F_{\pi,D} = F_{z,D} N_D^{-1} \quad (9)$$

(3) Simple rule: The Fed does not minimizes the loss function (5). One only seeks determinacy sets of rule parameters $F_{z,S}$ and $F_{\pi,S}$ for the *ad hoc*

rational expectations linear system including the private sector's law of motion (1, 2), boundary conditions of the forcing variable (3, 4), private sector inflation rule (8) and policy rule (10) instead of (9):

$$i_t = F_{z,S} z_t + F_{\pi,S} \pi_t \tag{10}$$

(4) Laissez faire equilibrium: It is a corner solution of the simple rule equilibrium where the rule parameters are equal to zero: $F_{z,S} = F_{z,S} = 0$:

$$i_t = 0 \tag{11}$$

The following sections details the solutions of each of these equilibria.

3 Optimal policy under commitment

The Fed chooses optimal policy while taking expected inflation law of motion as constraints. Her Lagrangian includes a Lagrange multipliers $2\beta^{t+1}\mu_{t+1}$.

$$\mathcal{L} = -\sum_{t=0}^{+\infty} \beta^t \left[\begin{array}{c} \pi_t^2 + 2Q_{\pi z}\pi_t z_t + R i_t^2 \\ +2\beta\mu_{t+1}\left[(1+\sigma)\pi_t + z_t - \sigma i_t - \pi_{t+1}\right] \end{array} \right] \tag{12}$$

Because of the certainty equivalence principle for determining optimal policy in the linear quadratic regulator including additive normal random shocks (Simon (1956)), the expectations of random variables ε_t are equal to zero and do not show up in the Lagrangian. In what follows, let us denote $\pi_{t+1} = E_t \pi_{t+1}$. The first order conditions are with respect to the Fed's target (forward inflation π_{t+1}) and with respect to the Fed's instrument (policy interest rate i_t):

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = 0 \Rightarrow \beta(1+\sigma)\mu_{t+1} = \mu_t - \pi_t - Q_{\pi z} z_t \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial i_t} = 0 \Rightarrow \beta(-\sigma)\mu_{t+1} = -R i_t \tag{14}$$

Substitute the Fed's Lagrange multiplier of inflation by the Fed's interest rate in the Fed's Euler equation $\frac{\partial \mathcal{L}}{\partial \pi_t}$. The *Fed's interest rate Euler equation* links recursively the future value of Fed's interest rate to its current value, because of the Fed's relative costs of changing interest rate $R > 0$ in its loss function:

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = 0 \Rightarrow \left(1 + \frac{1}{\sigma}\right) R i_t + \pi_t + Q_{\pi z} z_t = \frac{R}{\beta\sigma} i_{t-1} \tag{15}$$

The Hamiltonian system and its boundary conditions can be alternatively written the Euler equation including the policy rate or with the Euler equation with the Lagrange multiplier on inflation:

$$\left\{\begin{array}{c} z_t = \rho z_{t-1} + \varepsilon_t \\ E_t\pi_{t+1} = (1+\sigma)\,\pi_t - \sigma i_t + z_t \\ \left(1+\frac{1}{\sigma}\right) R i_t + \pi_t + Q_{\pi z} z_t = \frac{R}{\beta\sigma} i_{t-1} \\ \mu_{t+1} = \frac{R}{\beta\sigma^2} i_t \\ 0 = \lim_{t\to+\infty} \beta^t \mu_{\pi,t} \\ \frac{\partial L^*}{\partial\pi_0} = 0 = \mu_{\pi,t=0} \text{ and } z_0 \text{ given} \end{array}\right. \Leftrightarrow \left\{\begin{array}{c} z_t = \rho z_{t-1} + \varepsilon_t \\ E_t\pi_{t+1} = (1+\sigma)\,\pi_t - \frac{\beta\sigma^2}{R}\mu_{t+1} + z_t \\ \beta\,(1+\sigma)\,\mu_{t+1} = \mu_t - \pi_t - Q_{\pi z} z_t \\ i_t = \frac{\beta\sigma^2}{R}\mu_{t+1} \\ 0 = \lim_{t\to+\infty} \beta^t \mu_{\pi,t} \\ \frac{\partial L^*}{\partial\pi_0} = 0 = \mu_{\pi,t=0} \text{ and } z_0 \text{ given} \end{array}\right. \tag{16}$$

In the appendix, it is shown that the Hamiltonian system when $z_t = 0$ including inflation and the Lagrange multiplier of inflation has the usual saddlepoint equilibrium property: one eigenvalue is stable and the other eigenvalue is unstable. Hence, the full system including inflation, the Lagrange multiplier of inflation (or the policy rate) and the stationary dynamics of the forcing variable $z_t \neq 0$ with stable eigenvalue ρ includes two stable eigenvalue and one unstable eigenvalue. Using the infinite horizon transversality conditions, the relevant solution is the one that stabilizes the state-costate vector for any initialization of inflation π_0 and of the exogenous shock z_0 in a stable subspace of dimension two within a space of dimension three (z_t, π_t, μ_t) or (z_t, π_t, i_t) of the Hamiltonian system (Anderson *et al.* (1996)). We seek a characterization of the Lagrange multiplier μ_t on inflation minimizing the loss function L_t^* at all dates t such that it is a linear function of the two other variables in order to remain in the stable invariant subspace of dimension two of the Hamiltonian system:

$$\mu_t = -P_\pi \pi_t - P_{\pi z} z_t = \frac{\partial v\,(\pi_t, z_t)}{\partial \pi_t} \tag{17}$$

The parameter P_π of the optimal value v_0 is the solution of a Riccati equation. The parameter $P_{\pi z}$ of the optimal value v_0 is the solution of a Sylvester equation (Anderson et al. (1996)). Assuming $\beta = 1$ for simplicity in what follows, we obtain (see appendix):

$$P_\pi\left(\underset{-}{\sigma}, \underset{+}{R}\right) = \frac{1}{2} + R\left(\frac{1}{2}+\frac{1}{\sigma}\right) - \sqrt{\left(\frac{1}{2} + R\left(\frac{1}{2}+\frac{1}{\sigma}\right)\right)^2 + \frac{R}{\sigma^2}} \tag{18}$$

$$P_{\pi z}\left(\underset{-}{\sigma}, \underset{+}{R}, \underset{+}{\rho}, \underset{+}{Q_{\pi z}}\right) = \frac{Q_{\pi z} + (1+\sigma-\sigma F_\pi)\,P_\pi}{1-(1+\sigma-\sigma F_\pi)\,\rho}. \tag{19}$$

Alternatively, we seek a characterization of the Fed's policy rate such that it is a linear function of the two other variables in order to remain in the stable invariant subspace of dimension two of the Hamiltonian system, with the optimal rule parameters $F_{\pi,C}$ and $F_{z,C}$ solution of the discounted augmented linear quadratic regulator (see appendix):

$$i_t = F_{\pi,C}\pi_t + F_{z,C}z_t \tag{20}$$

$$F_{\pi,C}\left(\underset{-}{\sigma}, \underset{-}{R}\right) = \frac{\sigma P_\pi + \sigma^2 P_\pi}{R + \sigma^2 P_\pi} = \frac{1 + \frac{1}{\sigma}}{1 + \frac{1}{\sigma}\left(\frac{R}{\sigma P_\pi}\right)} > 1 \tag{21}$$

$$F_{z,C}\left(\underset{-}{\sigma}, \underset{-}{R}, \underset{-}{\rho}, \underset{+}{Q_{\pi z}}\right) = \frac{1}{\sigma}\frac{1 + \frac{\rho P_{\pi z}}{P_\pi}}{1 + \frac{R}{\sigma^2 P_\pi}} \tag{22}$$

The optimal initial anchor (or jump) of inflation on the forcing variable is found minimizing the policy maker's loss function, with the Lagrange multiplier of inflation μ_0 predetermined to the value zero:

$$\mu_0 = P_\pi \pi_0 + P_{\pi z} z_0 = 0 \Rightarrow \pi_0 = -\frac{P_{\pi z}}{P_\pi} z_0. \tag{23}$$

Alternatively, the value of the policy rate $i_{-1} = \frac{\beta\sigma^2}{R}\mu_0 = 0$ is predetermined at zero at the date before the optimization (there is no past promises to keep) and the value of the policy rate i_0 at the date of the optimization is given by:

$$i_0 = F_{\pi,C}\left(-\frac{P_{\pi z}}{P_\pi}\right) z_0 + F_{z,C} z_0. \tag{24}$$

If the Lagrange multiplier of inflation is predetermined at zero at the date $t = 0$ of optimization, the interest rate is predetermined at zero at the date $t = 0$ of optimization. Both players, including the private sector, know that the Lagrange multiplier of inflation (or the policy interest rate) is *predetermined*, because it is the costate of a forward variable (inflation).

A representation of optimal policy under commitment dynamics is given by the private sector law of motion (1,2), the optimal policy rule which is the substitute of the Fed's Euler interest rate once the infinite horizon transversality condition (7) has been taken into account and the optimal initial anchor of inflation (23):

$$(S_\pi)\begin{cases} z_t = \rho z_{t-1} + \varepsilon_t \\ E_t\pi_{t+1} = (1+\sigma)\,\pi_t - \sigma i_t + z_t \\ i_t = F_{\pi,C}\pi_t + F_{z,C}z_t \\ \pi_0 = -\frac{P_{\pi z}}{P_\pi} z_0 \text{ with } z_0 \text{ given} \\ \mu_t = -P_\pi\pi_t - P_{\pi z}z_t \\ \text{for all dates } t \geq 0. \end{cases} \Leftrightarrow (S_\mu)\begin{cases} z_t = \rho z_{t-1} + \varepsilon_t \\ E_t(-P_\pi^{-1}\mu_{t+1} - P_\pi^{-1}P_{\pi z}z_{t+1}) = \\ (1+\sigma)\,(-P_\pi^{-1}\mu_t - P_\pi^{-1}P_{\pi z}z_t) - \sigma i_t + z_t \\ i_t = -F_{\pi,C}P_\pi^{-1}\mu_t + (F_{z,C} - F_{\pi,C}P_\pi^{-1}P_{\pi z})z_t \\ \text{with } F_{\mu,z} = -F_{\pi,C}P_\pi^{-1} \\ \text{with } F_{\mu,z} = F_{z,C} - F_{\pi,C}P_\pi^{-1}P_{\pi z} \\ \mu_0 = -P_\pi\pi_0 - P_{\pi z}z_0 = 0 \text{ with } z_0 \text{ given} \\ \pi_t = -P_\pi^{-1}\mu_t - P_\pi^{-1}P_{\pi z}z_t \\ \text{for all dates } t \geq 0. \end{cases} \tag{25}$$

Backus and Driffill (1986) find the system (S_π) for optimal policy. Ljundqvist and Sargent (2012, chapter 19) begin with the system (S_π) and finally use the mathematically equivalent system (S_μ) for optimal policy. They found the system (S_μ) after linear substitution of inflation by its Lagrange multiplier and the forcing variable using the linear equation $\pi_t = -P_\pi^{-1}\mu_t - P_\pi^{-1}P_{\pi z}z_t$ in all the equations of the system (S_π). The system of equations (S_π) with a policy rule function of private sectors of private sector's predetermined and forward variables (z_t,π_t) and the system of equations (S_μ) with a policy rule function of predetermined variables (z_t,μ_t) are *mathematically and observationally equivalent*. The initial transversality condition $\mu_0 = -P_\pi\pi_0 - P_{\pi z}z_0 = 0$ with $P_\pi \neq 0$ in the system (S_μ) is *mathematically and observationally equivalent* to $\pi_0 = -\frac{P_{\pi z}}{P_\pi}z_0$ in the system (S_π).

The recursive dynamics after substitution of the optimal policy rule function of the initial conditions is given by:

$$\begin{pmatrix} E_t z_{t+1} \\ E_t\pi_{t+1} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-\sigma F_{z,C}}{\rho - \left(1+\sigma-\sigma F_{\pi,C}\right)} \end{pmatrix} \rho^t z_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1+\sigma-\sigma F_{\pi,C})^t \underbrace{\left(-\frac{P_{\pi z}}{P_\pi}z_0 - \frac{1-\sigma F_{z,C}}{\rho - (1+\sigma-\sigma F_{\pi,C})}z_0\right)}_{\text{usually} \neq 0} \tag{26}$$

The optimal inflation rule parameter $F_{\pi,C}\left(\underset{-}{\sigma}, \underset{-}{R}\right)$ decreases with the monetary transmission parameter σ and with the Fed's relative weight R on the volatility of the policy rate. When varying Fed's preference $0 < R < +\infty$ for a given monetary policy transmission parameter $\sigma > 0$, the inflation rule parameter $F_{\pi,C}\left(\underset{-}{\sigma}, \underset{-}{R}\right)$ varies within the following "Taylor principle" set $]1+\frac{1}{1+\sigma},1+\frac{1}{\sigma}[$: the inflation rule parameter is larger than one. The inflation growth factor λ_π is a linear decreasing function of the inflation rule parameter $F_{\pi,C}$. When hich varies between zero and the inverse of the open-loop growth factor for maximal inertia of the Fed ($R \rightarrow +\infty$): $]0,\frac{1}{1+\sigma}[$.

$$0 < R < +\infty_t \text{ and } \sigma > 0 \Rightarrow 1+\frac{1}{1+\sigma} < F_{\pi,C} < 1+\frac{1}{\sigma}$$
$$\Rightarrow 0 < \lambda_\pi = 1+\sigma-\sigma F_{\pi,C} < \frac{1}{1+\sigma} < 1$$

There are two stable eigenvalues ρ and $\lambda_\pi = 1+\sigma-\sigma F_{\pi,C}$. Expected inflation and expected forcing variable (π_t, z_t) dynamics of optimal policy under commitment is a converging sink with an initial optimal anchor (jump) of inflation. The number of stable roots (two) for the saddle point equilibrium of this Hamiltonian system is equal to the number (two) of predetermined variables: (z_t, μ_t) or (z_t, i_t) in the three dimensions space (π_t, z_t, μ_t) or (π_t, z_t, i_t).

Both players, including the private sector, know that the Lagrange multiplier of inflation (or the policy interest rate) is *predetermined*, because it is the costate of a forward variable (inflation). The Hamiltonian system with boundary conditions (S_π) including the representation of the feedback rule $i_t = F_{\pi,C}\pi_t + F_{z,C}z_t$ satisfies Blanchard and Kahn (1980) determinacy condition. The number of stable roots (2) for the saddlepath equilibrium of this Hamiltonian system which is equal to the number (2) of predetermined variables: (z_t, μ_t) or (z_t, i_t) in the three dimensions space (π_t, z_t, μ_t) or (π_t, z_t, i_t).

4 Discretionary policy

Oudiz and Sachs (1985) discretionary policy equilibrium assumes that the private sector does not believe in the policy maker's commitment. Both private and policy maker know that their best response rules only depend on predetermined variable at all periods (hence: $F_\pi = 0$) with constant rule parameters F_z and N_z to be chosen:

$$\pi_t = N_D z_t \quad (27)$$
$$i_t = F_{z,D} z_t \quad (28)$$

Substituting private sector inflation rule (8) and policy rule (9) in the inflation law of motion (1) and comparing it with the forcing variable law of motion (2) leads to the following relation beween N_D and $F_{z,D}$:

$$E_t z_{t+1} = \frac{1}{N_D}\left(z_t + (1+\sigma)N_D z_t - \sigma F_{z,D} z_t\right) = \left(1+\sigma+\frac{1-\sigma F_{z,D}}{N_D}\right) z_t = \rho z_t$$

$$\Rightarrow N_D = N - N\sigma F_{z,D} \text{ with } N = \frac{1}{\rho-(1+\sigma)} \quad (29)$$

Substituting private sector inflation rule (8) and policy rule (9) in the loss function (3) leads to the optimal program:

$$\max_{\{F_z,N_D\}} -\frac{1}{2}\left(N_{z,D}^2 + 2Q_{\pi z}N_D + RF_{z,D}^2\right)\frac{1}{1-\beta\rho^2}z_0^2 \quad (30)$$

$$N_D = N - N\sigma F_{z,D} \text{ with } N = \frac{1}{\rho-(1+\sigma)} \quad (31)$$

with solutions:

$$0 < R < +\infty \Rightarrow \quad (32)$$

$$0 < F_{z,D} = \frac{1}{\sigma}\frac{1+\frac{Q_{\pi z}}{N}}{1+\frac{R}{\sigma^2 N^2}} < \frac{1+\frac{Q_{\pi z}}{N}}{\sigma} \quad (33)$$

$$-Q_{\pi z} < N_D = N - N\frac{1+\frac{Q_{\pi z}}{N}}{1+\frac{R}{\sigma^2 N^2}} < N \quad (34)$$

Discretionary policy for all dates $t > 0$ has these two following representations related to two mathematically and observationally equivalent systems of equations:

$$(S_{D,z})\left\{\begin{array}{c} z_t = \rho z_{t-1} + \varepsilon_t \\ \pi_t = N_D z_t \\ N_D = \frac{1-\sigma F_{z,D}}{\rho-(1+\sigma)} \\ i_t = F_{z,D} z_t \\ z_0 \text{ given} \end{array}\right. \Longleftrightarrow (S_{D,\pi})\left\{\begin{array}{c} \pi_t = \rho \pi_{t-1} + N_D \varepsilon_t \\ z_t = \frac{1}{N_D}\pi_t \\ N_D = \frac{1}{\rho-\left(1+\sigma-\sigma F_{\pi,D}\right)} \\ i_t = F_{\pi,D}\pi_t \text{ with } F_{\pi,D} = \frac{F_{z,D}}{N_D} \\ z_0 \text{ given} \end{array}\right. \quad (35)$$

The discretionary inflation rule parameter $F_{\pi,D}$ varies in the following "Taylor principle set" which is strictly negative:

$$0 < R < +\infty \Rightarrow \frac{1}{\sigma N}\left(-\frac{N}{Q_{\pi z}} - 1\right) < F_{\pi,D} = \frac{1}{N}\frac{F_{z,D}}{1-\sigma F_{z,D}} < 0 \quad (36)$$

The discretionary Taylor principle is such that the inflation parameter of discretionary policy is strictly negative when varying Fed's preferences (R) for the relative cost of changing the policy rate:

$$\left\{\begin{array}{c} F_{\pi,D} \in \left]-\infty, 0\right[\\ \sigma > 0 \\ 0 < N_D < N \\ 0 < R < +\infty \end{array}\right. \Longleftrightarrow \left\{\begin{array}{c} \lambda_{\pi,D} = 1 + \sigma - \sigma F_{\pi,D} \in \left]1+\sigma, +\infty\right[\\ \sigma > 0 \\ 0 < N_D < N \\ 0 < R < +\infty \end{array}\right. \tag{37}$$

The expected dynamics function of initial conditions that out of equilibrium paths are diverging:

$$\leq \begin{pmatrix} E_t z_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{\rho - \left(1+\sigma-\sigma F_{\pi,D}\right)} \end{pmatrix} \rho^t z_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(1+\sigma-\sigma F_{\pi,D}\right)^t \underbrace{\left(-\frac{1}{\rho-\left(1+\sigma-\sigma F_{\pi,D}\right)} z_0 + \pi_0\right)}_{=0}$$

Finally, discretionary policy inflation π_t is exactly negatively correlated with the auto-regressive forcing variable z_t ($N_D < 0$), with a coefficient of determination equal to one. The policy rate i_t is exactly negatively correlated with inflation π_t ($F_{\pi,D} < 0$) or exactly positively correlated with the auto-regressive forcing variable z_t ($F_{z,D} > 0$) with coefficients of determination equal to one.

5 Simple rule

Simple rule equilibrium assume that the private sector does not believe in the policy maker commitment. The private sector policy rule depends only on predetermined variable at all periods as in Oudiz and Sachs (1985) discretionary policy (equation (8):

$$\pi_t = N_S z_t \tag{38}$$

The policy maker is myopic: she does not notice that his commitment is not credible for the private sector. Her simple policy rule depends on forward and predetermined variables:

$$i_t = F_{z,S} z_t + F_{\pi,S} \pi_t \tag{39}$$

Combining private sector's rule with policy maker's simple rule imply that the policy maker policy rule is observationally equivalent to representations of the policy rule which depends only on a number of variables equal to the number of predetermined variables. In particular, it is observationally equivalent to a policy rule which depends only on one variable. The policy maker's is not credible to commit to a policy modifying recursively expectations of inflation:

$$\left\{\begin{array}{c}\pi_t = N_S z_t \\ i_t = F_{z,S} z_t + F_{\pi,S} \pi_t\end{array}\right. \Longleftrightarrow \left\{\begin{array}{c}\pi_t = N_S z_t \\ i_t = \left(F_{z,S} + F_{\pi,S} N_S\right) z_t\end{array}\right. \tag{40}$$

$$\Longleftrightarrow \left\{\begin{array}{c}z_t = N_S^{-1} \pi_t \\ i_t = \left(\frac{F_{z,S}}{N_S} + F_{\pi,S}\right) \pi_t\end{array}\right. \tag{41}$$

A non-optimal simple rule equilibrium consists of equations (1,2,4,5,8,10). The combination of the private sector inflation rule as a linear function of the forcing variable (8) and of a policy interest rule function of two variables imply a linear identification restriction on the interest policy rule, which can be written as a function of only one variable. For the Taylor principle, the usual practice is to consider the policy representation as a function of inflation (identification restriction $F_{z,S} = 0$):

$$i_t = F_{\pi,S} \pi_t \tag{42}$$

There is one predetermined variable z_t and two forward variables: inflation π_t and the policy interest rate i_t. The policy rate is no longer a predetermined variable, as in optimal policy, because the initial transversality condition on private sector's forward variable (6) have been eliminated by the private sector inflation rule (8).

There is one stable eigenvalue (ρ, $|\rho| < 1$) related to the predetermined exogenous forcing variable z_t. Then, the second eigenvalue should be unstable $|1 + \sigma - \sigma F_\pi| > 1$ according to Blanchard and Kahn's (1980) determinacy condition for rational expectations *ad hoc* linear system of equations. This implies the following Taylor principle set:

$$|\lambda(F_\pi)| = |1 + \sigma - \sigma F_{\pi,S}| > 1 \Leftrightarrow F_{\pi,S} \in]-\infty, 1[\cup]1 + \frac{2}{\sigma}, +\infty[\tag{43}$$

Depending on the identifying restrictions on policy rule parameters, we have these following representation $S_{S,z}$ or $S_{S,\pi}$ of the rational expectations simple rule unique solution with the policy rule and the VAR(1) function of the forcing variable z_t or of inflation π_t:

$$(S_{S,z})\begin{cases} z_t = \rho z_{t-1} + \Sigma_\varepsilon \varepsilon_t \\ \pi_t = N_S z_t \\ N_S = \frac{1-\sigma F_{z,S}}{\rho-(1+\sigma)} \\ i_t = F_{z,s} z_t \\ z_0 \text{ given} \end{cases} \Longleftrightarrow (S_{S,\pi})\begin{cases} \pi_t = \rho \pi_{t-1} + N_S \Sigma_\varepsilon \varepsilon_t \\ z_t = N_S^{-1} \pi_t \\ N_S = \frac{1}{\rho-\left(1+\sigma-\sigma F_{\pi,S}\right)} \\ i_t = F_{\pi,S} \pi_t \\ z_0 \text{ given} \end{cases} \quad (44)$$

6 Laissez faire equilibrium

Laissez faire equilibrium is a simple rule equilibrium assuming the simple rule inflation parameter $F_{\pi,S}$ is zero:

$$i_t = F_{\pi,S} \pi_t = 0 \text{ with } F_{\pi,S} = 0 \quad (45)$$

The optimal laissez-faire rational expectations equilibrium path can be written for all dates $t > 0$ with the two following representations related to two mathematically and observationally equivalent systems of equations:

$$\begin{cases} z_{t+1} = \rho z_t + \varepsilon_{t+1} \\ \pi_t = N z_t \\ N = \frac{1}{\rho-(1+\sigma)} < 0 \\ i_t = 0 \\ z_0 \text{ given} \end{cases} \Longleftrightarrow \begin{cases} \pi_{t+1} = \rho \pi_t + N \varepsilon_{t+1} \\ z_t = N^{-1} \pi_t \\ N = \frac{1}{\rho-(1+\sigma)} < 0 \\ i_t = 0 \\ z_0 \text{ given} \end{cases} \quad (46)$$

with the following constraint on the private sector's inflation rule parameter N:

$$0 < \rho < 1 \text{ and } \sigma > 0 \Rightarrow -\frac{1}{\sigma} < N = \frac{1}{\rho - (1+\sigma)} < -\frac{1}{1+\sigma} < 0$$

Laissez-faire inflation π_t is exactly negatively correlated with the auto-regressive forcing variable z_t ($N < 0$), with a coefficient of determination equal to one. The slope $N(\rho, \sigma)$ of the eigenvector of the stable exogenous eigenvalue ρ parameter decreases with the auto-correlation parameter ρ of the forcing variable. It increases with the (out of equilibrium) expected inflation growth rate $\sigma > 0$. One estimates the parameter ρ of an AR(1) model of inflation and the variance of the residuals $N^2\Sigma_\varepsilon^2$ increases with Σ_ε^2 and with σ and decreases with ρ. If the forcing variable z_t is not observable, then one cannot regress $\pi_t = N z_t$ and estimate directly the reduced form parameter N and the structural parameter σ. Then identifying Σ_ε^2 and σ requires an identification restriction.

7 Taylor principles

Table 1 sums up distinct Taylor principles. To have a *countercyclical negative feedback* inflation Taylor rule parameter, two predetermined variables are needed. Besides the forcing variable, either inflation is predetermined or inflation is forward and the Lagrange multiplier of inflation is predetermined (alternatively the policy interest rate) for optimal policy. Optimal policy under commitment is an *active* (or reactive) policy, with countercyclical credible anchor and the monitoring of private sector expectations dynamics. To have a *procyclical positive feedback* inflation Taylor rule parameter, one predetermined variable only is required. This is the case of discretionary policy and of simple rule. Because the private sector does not believe that the policymaker is credible for not reneging commitment, the policymaker is *unable* to monitor private sector's inflation expectations. For this reason, procyclical Taylor rule parameter is a *passive* discretionary monetary policy. The relation between the controllable eigenvalue depending on the inflation rule parameter is $\lambda_{\pi}(\sigma, F_{\pi}) = 1 + \sigma - \sigma F_{\pi}$. There is a large gap between the largest value of the inflation rule parameter with discretionary policy (zero) to its smallest value with optimal policy $1 + \frac{1}{1+\sigma}$. It is not feasible to have a continuous shift of policy rule parameters from one equilibrium to the other.

Table 1: Taylor principles

Policy	Predetermined inflation π_t	Forward inflation π_t
Optimal	2 predetermined: z_t, π_t 1 forward: i_t $R \in]0, +\infty[$ $F_\pi \in \left]1 + \frac{1}{1+\sigma}, 1 + \frac{1}{\sigma}\right[$ $\lambda_\pi \in \left]0, \frac{1}{1+\sigma}\right[$ countercyclical active policy	2 predetermined: z_t, i_t or μ_t 1 forward: π_t $R \in]0, +\infty[$ $F_{\pi,C} \in \left]1 + \frac{1}{1+\sigma}, 1 + \frac{1}{\sigma}\right[$ $\lambda_{\pi,C} \in \left]0, \frac{1}{1+\sigma}\right[$ countercyclical active policy
Discretionary	2 predetermined: z_t, π_t 1 forward: i_t $R \in]0, +\infty[$ $F_\pi \in \left]1 + \frac{1}{1+\sigma}, 1 + \frac{1}{\sigma}\right[$ $\lambda_\pi \in \left]0, \frac{1}{1+\sigma}\right[$ countercyclical active policy	1 predetermined: z_t 2 forward: π_t, i_t $R \in]0, +\infty[$ $F_{\pi,D} \in]-\infty, 0[\, , F_z = 0$ $\lambda_{\pi,D} \in]1 + \sigma, +\infty[$ procyclical passive policy
Simple rule	2 predetermined: z_t, π_t 1 forward: i_t $\lambda_\pi \in]-1, 1[$ $F_\pi \in \left]1, 1 + \frac{2}{\sigma}\right[$ countercyclical reactive policy	1 predetermined: z_t 2 forward: π_t, i_t $\lambda_{\pi,S} \in]1, +\infty[\cup]-\infty, -1[$ $F_{\pi,S} \in]-\infty, 1[\cup \left]1 + \frac{2}{\sigma}, +\infty\right[$ $F_{z,S} = 0$ procyclical passive policy

8 Revisiting Svensson vs McCallum and Nelson controversy

In table 2, numerical simulations track the relation between policy rules and the Fed relative cost of changing the policy rate. They allow to revisit part of the Svensson/McCallum controversy between optimal policy versus simple rule.

In the simulations, the transmission mechanism parameters correspond to the large persistence usually found for inflation $\rho = 0.9$ and a modest marginal effect of the user cost channel of monetary policy: a rise of nominal policy rate of 1% leads to a modest fall of next period inflation $-\sigma \cdot 1\% = -0.1\%$. The initial negative shock $z_0 = -10\%$ remains in the range of a assuming that the private sector's transmission mechanism equations (1 and 2) are a linear approximation of non-linear models. The shock is chosen to be negative in order to check when the nominal interest rate crosses the zero lower bound constraint set at -2% for a long run equilibrium inflation target set at 2%. More precisely, the response of inflation is positive. The negative correlation comes from the sign of the initial anchor of inflation on the forcing variable which is the sign of $\frac{1}{\rho-(1+\sigma)} = -5 < 0$. Then, the feedback response of the policy rate is negative. The laissez-faire equilibrium leads to a very large initial jump of inflation $\pi_0 = \frac{1}{\rho-(1+\sigma)} z_0 = 50\%$. For example, the initial jump could be reduced to the value $\pi_0 = 25\%$ in setting $z_0 = -5\%$ or in decreasing the inflation persistence $\rho = 0.8$ and in increasing the marginal effect of the policy rate on future inflation to $\sigma = 0.2$.

The policy preferences includes a zero discount rate and a discount factor equal to one: $\beta = 1$, a relative weight of inflation variance standardized to one: $Q_{\pi\pi} = 1$, a modest relative weight on the covariance of inflation with the forcing variable $Q_{\pi z} = 10\%$ of the weight on inflation variance, and the relative weight of policy rate variance measuring Fed's inertia varies in the range $R \in]0, +\infty[$.

Table 2: Differences between optimal policy, discretionary policy and simple rule: $\sigma = 0.1$, $\rho = 0.9$, $\beta = 1$, $Q_{\pi\pi} = 1$, $Q_{\pi z} = 0.1$ for an initial negative shock $z_0 = -0.1$.

$F_{\pi,S}$	R	$F_{\pi,D}$	$F_{\pi,C}$	$\lambda_{\pi,S}$	$\lambda_{\pi,D}$	$\lambda_{\pi,C}$	$F_{z,C}$
$-\infty$	$\rightarrow 0$	$-\infty$	11	$+\infty$	$+\infty$	0	11
-81.7	10^{-3}	-81.7	10.09	9.267	9.267	0.091	10.8
-4.67	0.1	-4.67	3.70	1.567	1.567	0.730	9.98
-0.49	1	-0.49	2.26	1.149	1.149	0.874	9.66
-0.05	10	-0.05	1.95	1.105	1.105	0.905	9.56
0	$\rightarrow +\infty$	0	1.91	1.1	1.1	0.905	9.56
0.9999	simple non-D	-	-	1.00001	-	-	-
21.001	simple non-D	-	-	-1.0001	-	-	-
30	simple non-D	-	-	-1.9	-	-	-
10000	simple non-D	-	-	-998.9	-	-	-

Figure 1 plots rule parameters $F_{\pi,C}$, $F_{\pi,D}$, $F_{z,C}$ as non linear functions of the relative cost of changing the policy rate R. Figure 2 plots eigenvalues $\lambda_{\pi,C}$ and $\lambda_{\pi,D}$ as non linear functions of the relative cost of changing the policy rate R.

As seen in table 1, for optimal policy under commitment, the inflation rule parameter is larger than one, positive and bounded $F_{\pi,C} \in \left]1 + \frac{1}{1+\sigma}, 1 + \frac{1}{\sigma}\right[=]1.91, 11[$. When the cost of changing the policy rate R increases from zero to infinity, optimal monetary policy is less reactive: the inflation rule parameter *decreases* from 11 to 1.91. The relation between the controllable eigenvalue and the inflation inflation rule parameter is $\lambda_\pi = 1+\sigma-\sigma F_\pi$. The eigenvalue λ_π increases from zero (fastest convergence) to the slowest optimal convergence with the inverse $\lambda_\pi = \frac{1}{1+\sigma} = 0.905$ of the laissez-faire value of λ_π equal to $1 + \sigma$.

For discretionary policy, the inflation rule parameter $F_{\pi,D}$ is negative. When the cost of changing the policy rate R increases from zero to infinity, the inflation rule monetary policy *increases* from minus infinity to zero. The eigenvalue λ_{π} is always larger than one. It increases $1+\sigma$ (laissez-faire equilibrium) to infinity. The eigenvalue λ_{π} has only an impact on the paths which are out of discretionary equilibrium, with faster divergence when there is a smaller jump of inflation in the discretionary equilibrium.

Finally, there is always a second rule parameter for optimal policy under commitment $F_{z,C}$ which is strictly different from zero, with rule parameters decreasing from 11 to 9.56 when the cost of changing the policy rate R increases from zero to infinity.

The following numerical simulation allows to revisit the Svensson/McCallum controversy between optimal policy versus simple rule:

(1) For given policy transmission mechanism parameters (ρ,σ), a given simple rule with negative inflation rule parameter $F_{\pi,S} \in \left]-\infty, 0\right[$ is always the *reduced* form of the rule of *discretionary* policy with inflation parameter $F_{\pi,D}$ with a unique Fed's preference parameter for the relative cost of chaning the policy rate $R = R\left(F_{\pi,S}\right)$. This result holds in this case because these simple rules includes a number of parameters which is equal to the number of identified parameters in a policy rule of discretionary policy, which is exactly the number of predetermined variables.

(2) By contrast, given simple positive policy rule parameters $F_{\pi,S} \in \left]0, 1\right[\cup \left]1+\frac{2}{\sigma}, +\infty\right[$ are never the reduced form of a rule of discretionary policy (raws "simple non-D" in table 2).

(3) Simple rules are never related to optimal policy under commitment $F_{\pi,S} \in \left]1+\frac{1}{1+\sigma}, 1+\frac{1}{\sigma}\right[$.

(4) The relation between Fed's preference parameter for inertia (relative cost of changing the policy rate) and the Taylor rule parameter is highly non linear with two asymptotes.

Points 1 and 2 seems to contradict both Svensson and McCallum in suggesting that the controversy is *not relevant* under three conditions: (A) the monetary transmission channel and the Fed preferences parameters do not change during the period under study, (B) the number of identified parameters of a simple rule is exactly equal to the number of pretermined variable, (C) the simple rule parameters belongs to set of parameters of discretionary policy when varying preferences.

Point 2 seems to contradict McCallum. Some values of simple rule are never compatible with (sub)-optimal outcomes of the discretionary equilibrium. As seen in table 3: $F_{\pi,S} \in \left]0, 1\right[$ leads to simple rule with determinacy which increases inflation and its initial jump with respect to laissez-faire. *A set of simple rule parameter values do not avoid "disasters" whereas discre-*

tionary policy avoids them.

Points 3 seems to contradict McCallum in the sense that *simple rules under commitment are never related to credible monetary policy under commitment.* The Fed commit to a simple rule, but the private sector never considers that the Fed is credible to monitor inflation expectations. This seems to contradict anecdotal evidence, at least during Volcker's mandate at the FEd.

Point 4 seems to contradict Svensson who suggested that the Central Bank should not disclose her policy rule parameters $\mathbf{F}$ but instead her preference parameters $(\mathbf{R}, \mathbf{Q})$ without necessarily an explicit disclosure of transmission mechanism parameters (ρ,σ). Missing information and the non-linear relationship between preferences and rule parameters blurs the transparency advocated by Svensson. Announcing Fed's preferences *and* Fed rule parameters is useful.

The controversy deals with other issues, such as robustness to model misspecification. Giordani and Söderlind (2004, appendix D, proposition 1) demonstrated that simple rule equilibrium is not robust to misspecification (in the Hansen and Sargent (2008) definition). "*An evil agent who is able to commit will choose a non-stationary (ever increasing or decreasing)* "non zero deviation" in the law of motion*, which makes the loss function unbounded... The misspecification feared is then a trend increase (or decrease) of inflation"* [p.2388]. Not only the policy maker who commits to a simple rule is not credible to monitor expectations, but also, he is "*defenceless against the evil agent*" when fearing misspecification of the monetary policy transmission channel. McCallum and Nelson (2005) definition of robustness is to search for simple instrument rule "*that performs at least moderately well -avoiding disasters- in a variety of models*", assuming simultaneously that, in forward-looking models, a "*good little devil*" agent *never* selects out of equilibrium trend increase of inflation (disasters). For this reason, McCallum and Nelson (2005) definition of robustness is *less* robust than Hansen and Sargent (2008) robustness.

Giordani and Söderlind (2004) proposed a robust discretionary equilibrium. It is based on the assumption that the evil agent excludes the possibility that there are heterogeneous beliefs on the credibility of policy makers. If ever there is a small proportion of private sectors agent who believe in the credibility of the policy maker, their inflation rule may depend on past or future values of inflation: $\pi_t = N_S z_{t-1} + M\pi_{t-1}$. In this case, it is likely that the aggregate evil agent is able to commit and to choose a non-stationary trend increase of inflation. In this case, policymaker's robust response is to shift from robust discretionary policy equilibrium to robust optimal policy under commitment.

9 Conditions for a Negligible Cost of Time Inconsistency

For large or small Fed's preference for inertia (R), in particular, *for the observed high inertia of the Fed with tiny variations of the nominal policy rate within the range* [0%,4%] *during the great moderation, the cost of time-inconsistency and, hence, the requirement of commitment are negligible for optimal policy.* Optimal policy is then a near-time-consistent policy. A near-time-consistent optimal policy and a time-consistent sub-optimal discretionary policy are to be chosen by policymaker's. Their key difference is opposite countercyclical versus procyclical inflation rule parameters and the size of the stable subspace of the economy.

Table 3 presents welfare, initial inflation and policy rate values for optimal policy under commitment, discretionary policy and simple rules, when varying the relative cost of changing the interest rate in the range $R \in]0, +\infty[$.

Table 3: Optimal policy, discretionary policy and simple rule: transmission mechanism: $\sigma = 0.1$, $\rho = 0.9$, Preferences: $\beta = 1$, $Q_{\pi\pi} = 1$, $Q_{\pi z} = 0.1$ for $z_0 = -10\%$.

$F_{\pi,S}$	R	$\frac{v_C^*}{v_{LF}^*}$	$\frac{v_D^*}{v_{LF}^*}$	$\frac{\pi_{0,C}^*}{\pi_{0,LF}^*}$	$\frac{\pi_{0,,D,S}^*}{\pi_{0,LF}^*}$	$\max \frac{i_{t,C}}{z_0}$	$\max \frac{i_{t,D,S}}{z_0}$
$-\infty$	$\rightarrow 0$	0	0	0	0	$10 = \frac{1}{\sigma}$	$10 = \frac{1}{\sigma}$
-81.7	10^{-3}	0.03	0.03	0.04	0.02	8.81	9.76
-4.67	0.1	0.23	0.28	0.44	0.30	4.27	7
-0.49	1	0.70	0.78	0.82	0.80	1.262	1.96
-0.05	10	0.98	0.98	0.98	0.98	0.172	0.239
0	$\rightarrow +\infty$	1	1	1	1	0	0
0.9999	simple	-	-	-	2	-	9.998
21.001	simple	-	-	-	-0.10	-	11.053
$+\infty$	simple	-	-	-	0	-	$10 = \frac{1}{\sigma}$

In the third and fourth column are presented the Fed's loss function for commitment v_C^* and for discretion v_D^* divided by laissez-faire loss function, which are decreasing function of the cost of changing the policy rate. An increase of the relative cost of changing the policy rate imply a decrease of welfare, a decrease of the interest rate, and a higher initial jump of inflation. For high and minimal inertia, welfare, the initial jump of inflation and the initial value of the policy rate are identical for commitment and for discretion. Large difference between optimal welfare and discretionary welfare (10%) occurs for a relative cost of changing the policy rate R between 5% and 3 times the relative cost of inflation deviation from target in the loss function (figure 3). The zero lower bound variations within the range [0%,4%] for shocks as high as 10% on the forcing variable z_0 are related to value of R

larger than one for optimal policy and R larger than 5 for discretionary policy (figure 4).

As seen in phase diagram 9, optimal policy benefits from the opportunity to control the eigenvalue λ_π downward and of the determinacy of the optimal initial anchor to have a path which is not restricted to a line, as all the other paths reported on figure 9. The path evolves in a two dimensions stable subpace. It exhibit a curvature, which shows that optimal policy decrease earlier inflation than with path constrained to remain on a straight line. For a given loss function, discretionary path is the best approximation of optimal policy being restricted to evolve in a stable subspace of lower dimension (dimension 1). As both paths have to satisfy long term final transversality conditions, both of them ends in the origin. Because of the accelerating decrease of inflation (curvature of optimal path), discretionary policy has to provide an initial lower jump of inflation than optimal policy, knowing that in the future, optimal policy will outperform discretionary policy. This is shown also in the impulse responses (figure 8). This is matched with a higher interest rate at the beginning of the period for discretionary policy with respect to optimal policy.

The mechanism of this fall of the inflation is the opposite of the negative feedback intuition, with very poor economic insight. This higher interest rate is related to a smaller slope of stable eigenvector, due to an increase of the unstable eigenvalue, with a lower negative value of the positive feedback inflation rule parameter.

The initial inflation jump π_0 (figure 6) and the eigenvalue λ_π (figure 7) are plotted as a function of the inflation rule parameters F_π as those two variables depend both on the cost of changing the policy rate R.

- An increase of the inflation rule parameter of optimal policy from 1.9 to 11 leads to a decrease of the jump from the value of laissez-faire to instantaneous adjustment of inflation. The out of equilibrium paths (in figure 10,11,12, two of them are represented with deviation from the initial jump of $\pm 5\%$) are converging faster and fastor to the optimal path which suggests an increase of the costs of misspecification.

- A decrease of the inflation rule parameter of discretionary policy from zero to minus infinity leads to a decrease of the jump from the value of laissez-faire to instantaneous adjustment of inflation. Neighbouring out of equilibrium paths are more and more diverging (in figure 10,11,12, two of them are represented with deviation from the initial jump of $\pm 5\%$), which suggests an increase of the costs of misspecification.

- Inflation simple rule parameters between zero and one overshoot inflation up to doubling the value of the initial shock when reaching one. Inflation simple rule parameters over 21 overshoot inflation down with an opposite sign of the jump with respect to laissez faire. This overshooting is also seen in impulses response functions (figure 8) and phase diagram (figure 9).

- With old Keynesian models, there is no jump. But only a subset of stabilizing values of the rule parameters are optimal (between 1.95 and 11), the remaining values between 1 and 1.95 and between 11 and 21 could be used in simple rule old Keynesian models.

- Timeless perspective assume that the optimal jump occured for example 20 periods before. It is not clear whether it is related to the shock z_0 of now or if the shock occured twenty periods ahead z_{-20} . This is equivalent to consider as the current inflation jump the value of inflation 20 periods ahead. Inflation jump is minimal and very close to the long run value of inflation. Timeless perspective optimal policy is equivalent to optimal policy with a very low cost R of changing the policy rate and maximal volatility of the policy rate. In this case, optimal policy and discretionary policy are equivalent, so time-consistency does not matter any more. There is no cost of reneging commitment. A main issue is that timeless perspective may be applied to high cost R of changing the policy rate (which is also time-consistent with minimal volatility of the policy rate). A minimal volatility of the policy rate is cannot lead to a minimal jump of inflation in optimal policy. In this case, timeless perspective is useless perspective.

For visualizing time-inconsistency problem in phase diagrams, figures 16,17,18 plots phase diagrams of t=0 optimal policy under commitment, t=3 optimal policy under commitment in the plane with inflation and the forcing variable.

(1) For intermediate values of the cost of changing the policy rate (figure 14),The straight line joining the optimal jump to the origin would be the paths chosen if the policymaker optimizes again at each period and if nonetheless the private sector believes the policy maker is credible. It is defined as $\pi_t = -P_{\pi z} P_{\pi}^{-1} z_t$. This straight line is not equal to the discretionary path, because the discretionary path leads to another value of the initial jump $\pi_t = N_{z,D} z_t$. A second curve corresponds to the $t = 3$ optimal policy. The policy maker will anchor initially inflation at $\pi_3 = -P_{\pi z} P_{\pi}^{-1} z_3$ and renege her commitment from the path of $t = 0$ optimal policy. After several periods, $t = 3$ optimal policy path joins the $t = 0$ optimal policy path.

(2) For very large of very small cost of changing the policy rate ($R = 10$ or $R = 10^{-3}$), $t = 0$ optimal policy and $t = 3$ optimal policy follows nearly the same straight line, so that reneging commitment implies negligible or second order cost of time inconsistency. The paths of optimal policy under commitment and of discretionary policy are the same, but their rule countercyclical versus procyclical rule parameters are different.

10 Conclusion

There are distinct Taylor principles for optimal policy under commitment and discretionary policy. *Although the paths of both policies and hence the welfare are nearly identical for very inertial or non-inertial behaviour of the Fed* (large or negligible relative cost of changing the interest rate with respect to the cost of deviating from inflation target), *the policy rule recommandation are the opposite: procyclical rule parameters with discretionary policy and countercyclical rule parameters with optimal policy under commitment.* Finding distinct sets of policy rule parameters function of the private sectors variables for optimal policy under commitment versus discretionary policy can be done for any DSGE model of the private sector.

Further work may investigate empirical issues. The largest size of the stable subspace of optimal policy under commitment with respect discretionary policy has implications for applied econometrics. The number of endogenous stationary variables increases in the vector autoregressive component of the rational expectations system. Hence, it is no longer necessary to add AR or ARMA components to shocks for each of the forward variables in order to fit the data.

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11 Appendix: Augmented Linear Quadratic Regulator

For the augmented discounted quadratic regulator, write the system in matrix form:

$$\begin{pmatrix} z_{t+1} \\ \pi_{t+1} \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 1 & 1+\sigma \end{pmatrix} \begin{pmatrix} z_t \\ \pi_t \end{pmatrix} + \begin{pmatrix} 0 \\ -\sigma \end{pmatrix} i_t + \begin{pmatrix} \sigma_\varepsilon \\ 0 \end{pmatrix} \varepsilon_t \qquad (47)$$

To match Anderson et al. (1996) notations, define elements of matrices **A** and **B** as:

$$\begin{pmatrix} \rho & 0 \\ 1 & 1+\sigma \end{pmatrix} = \begin{pmatrix} A_{zz} & 0 \\ A_{\pi z} & A_{\pi\pi} \end{pmatrix}, \begin{pmatrix} 0 \\ B_{\pi} \end{pmatrix} = \begin{pmatrix} 0 \\ -\sigma \end{pmatrix} \tag{48}$$

$$A_{zz} = \rho, A_{\pi z} = 1, A_{\pi\pi} = 1+\sigma, B_{\pi} = -\sigma \tag{49}$$

In section 3, optimal rule parameters $\mathbf{F}_{\pi}$ and $\mathbf{F}_z$ and weights $\mathbf{P}_{\pi}$ and $\mathbf{P}_z$ of the optimal value of the loss function are found using linear substitions in the Hamiltonian system (Anderson et al. (1996)) leading to there formulas for the augmented linear quadratic regulator (we assume $\beta = 1$):

$$\mathbf{P}_{\pi} = \mathbf{Q}_{\pi\pi} + \beta\mathbf{A}'_{\pi\pi}\mathbf{P}_{\pi}\mathbf{A}_{\pi\pi} - \beta\mathbf{A}'_{\pi\pi}\mathbf{P}_{\pi}\mathbf{B}_{\pi}\left(\mathbf{R} + \beta\mathbf{B}'_{\pi}\mathbf{P}_{\pi}\mathbf{B}_{\pi}\right)^{-1}\beta\mathbf{B}'_{\pi}\mathbf{P}_{\pi}\mathbf{A}_{\pi\pi} \tag{50}$$

$$\mathbf{F}_{\pi,C} = \left(\mathbf{R} + \beta\mathbf{B}'_{\pi}\mathbf{P}_{\pi}\mathbf{B}_{\pi}\right)^{-1}\beta\mathbf{B}'_{\pi}\mathbf{P}_{\pi}\mathbf{A}_{\pi\pi} \tag{51}$$

$$\mathbf{P}_z = \mathbf{Q}_{\pi z} + \beta\left(\mathbf{A}_{\pi\pi} + \mathbf{B}_{\pi}\mathbf{F}_{\pi,C}\right)'\mathbf{P}_{\pi}\mathbf{A}_{\pi z} + \beta\left(\mathbf{A}_{\pi\pi} + \mathbf{B}_{\pi}\mathbf{F}_{\pi,C}\right)'\mathbf{P}_z\mathbf{A}_{zz} \tag{52}$$

$$\mathbf{F}_{z,C} = \left(\mathbf{R} + \beta\mathbf{B}'_{\pi}\mathbf{P}_{\pi}\mathbf{B}_{\pi}\right)^{-1}\beta\mathbf{B}'_{\pi}\left(\mathbf{P}_{\pi}\mathbf{A}_{\pi\pi} + \mathbf{P}_z\mathbf{A}_{zz}\right) \tag{53}$$

In what follows, it is shown that the Hamiltonian system includes a stable subspace of dimension 2 in a three dimension space defined by the three variables (z_t, π_t, μ_t). The Fed's Lagrangian system includes stationary autoregressive forcing shock law of motion, expected inflation law of motion (the two equations describing the private sector) and the first order condition on Fed's interest rate (or on Fed's Lagrange multiplier on inflation) linking its current optimal value to its next period optimal value (*the Euler equation of Fed's interest rate*):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \beta\frac{\sigma^2}{R} \\ 0 & 0 & \beta(1+\sigma) \end{pmatrix}\begin{pmatrix} z_{t+1} \\ \pi_{t+1} \\ \mu_{t+1} \end{pmatrix} = \begin{pmatrix} \rho & 0 & 0 \\ 1 & 1+\sigma & 0 \\ -Q_{\pi z} & -1 & 1 \end{pmatrix}\begin{pmatrix} z_t \\ \pi_t \\ \mu_t \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} z_{t+1} \\ \pi_{t+1} \\ \mu_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} \rho & 0 & 0 \\ 1 + \frac{\sigma^2}{R(1+\sigma)}Q_{\pi z} & 1+\sigma+\frac{\sigma^2}{R(1+\sigma)} & -\frac{\sigma^2}{R(1+\sigma)} \\ -\frac{1}{\beta(1+\sigma)}Q_{\pi z} & -\frac{1}{\beta(1+\sigma)} & \frac{1}{\beta(1+\sigma)} \end{pmatrix}}_{=\mathbf{M}^a}\begin{pmatrix} z_t \\ \pi_t \\ \mu_t \end{pmatrix}$$

Using the infinite horizon transversality conditions, the relevant solution is the one that stabilizes the state-costate vector for any initialization of inflation π_0 and of the exogenous shock z_0 in a stable subspace of dimension two within a space of dimension three (π_t, μ_t, z_t) of the Lagrange system (Anderson *et al.* (1996)). We seek a characterization of the Lagrange multiplier μ_t on inflation minimizing the optimal value function $v(\pi_t, z_t)$ at all date t of the form:

$$v(\pi_t, z_t) = -\frac{1}{2}\left(P_\pi \pi_t^2 + 2P_{\pi z}\pi_t z_t\right) \text{ with } P_\pi \neq 0 \quad (54)$$

$$\frac{\partial v(\pi_t, z_t)}{\partial \pi_t} = -P_\pi \pi_t - P_{\pi z} z_t = \mu_t \quad (55)$$

such that the resulting sequence (z_t, π_t, μ_t) is in the stable subspace of the augmented matrix $\mathbf{M}^a$. *When the forcing sequence is initialized at zero* $z_0 = 0$, the matrix P_π is such that all vectors of the stable subspace of the matrix $\mathbf{M}$ related to the controllable part of the system can be represented as $(\pi_t, P_\pi \pi_t)$. First, consider the solution when the forcing sequence is initialized at zero $z_0 = 0$ in order to find the policy instrument optimal response parameter F_π to the endogenous variable. Second, compute the policy instrument optimal response parameter F_z to the non-controllable forcing variable. For the first step, the optimal system is not augmented:

$$\begin{pmatrix} \pi_{t+1} \\ \mu_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1+\sigma+\frac{\sigma^2}{R(1+\sigma)} & -\frac{\sigma^2}{R(1+\sigma)} \\ -\frac{1}{\beta(1+\sigma)} & \frac{1}{\beta(1+\sigma)} \end{pmatrix}}_{=\mathbf{M}} \begin{pmatrix} \pi_t \\ \mu_t \end{pmatrix}$$

The Fed's Lagrange system includes one eigenvalue λ_1 with absolute value below one and the other eigenvalue is such that $\lambda_2 = 1/\lambda_1$ because the matrix $\mathbf{M}$ is symplectic (Anderson et al. (1996)). The Jordan transform and the transversality condition at the final date leads to Blanchard and Kahn's (1980) unique stable solution ($P_{\lambda_1^{-1}}$ is the slope of the eigenvectors of the unstable eigenvalue $1/\lambda_1$):

$$\begin{pmatrix} \pi_{t+1} \\ \mu_{t+1} \end{pmatrix} = \begin{pmatrix} 1 \\ P_\pi \end{pmatrix} \lambda_1^t \left(\frac{\mu_0 - P_{\lambda_1^{-1}}\pi_0}{P_\pi - P_{\lambda_1^{-1}}}\right) + \begin{pmatrix} 1 \\ P_{\lambda_1^{-1}} \end{pmatrix} \lambda_1^{-t} \left(\frac{P_\pi \pi_0 - \mu_0}{P_\pi - P_{\lambda_1^{-1}}}\right) \quad (56)$$

$$\lim_{t\to+\infty} \beta^{t+1}\mu_{t+1} = 0, \text{ transversality condition for } t \to +\infty. \quad (57)$$

$$\begin{pmatrix} \pi_{t+1} \\ \mu_{t+1} \end{pmatrix} = \begin{pmatrix} 1 \\ P_\pi \end{pmatrix} (1+\sigma-\sigma F_\pi)^t \pi_0 \text{ with } \mu_t = P_\pi \pi_t \text{ for all } t \in \mathbb{N}.$$

The policy maker's transversality condition at the final date (infinite horizon) rules out diverging paths driven by the unstable eigenvalue λ_1^{-1}. Hence, the optimal path is of dimension one, driven by converging powers of the unique stable eigenvalue λ_1^t, along the stable arm of the two dimensions saddlepoint equilibrium.

Finally, one obtains stable dynamics function of initial conditions with the representation using these two private sector's variables (z_t, π_t) among the set of three variables (z_t, π_t, i_t) or (z_t, π_t, μ_t):

$$\begin{pmatrix} E_t z_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \left(\begin{pmatrix} \rho & 0 \\ 1 & 1+\sigma \end{pmatrix} + \begin{pmatrix} 0 \\ -\sigma \end{pmatrix} \begin{pmatrix} F_z^* & F_\pi^* \end{pmatrix} \right) \begin{pmatrix} z_t \\ \pi_t \end{pmatrix} \tag{58}$$

$$\begin{pmatrix} E z_{t+1} \\ E \pi_{t+1} \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 1-\sigma F_z^* & 1+\sigma-\sigma F_\pi^* \end{pmatrix} \begin{pmatrix} z_t \\ \pi_t \end{pmatrix} \tag{59}$$

$$\begin{pmatrix} E_t z_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1-\sigma F_z^*}{\rho-(1+\sigma-\sigma F_\pi^*)} & 1 \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & 1+\sigma-\sigma F_\pi^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1-\sigma F_z^*}{\rho-(1+\sigma-\sigma F_\pi^*)} & 1 \end{pmatrix} \begin{pmatrix} z_t \\ \pi_t \end{pmatrix} \tag{60}$$

$$\begin{pmatrix} E_t z_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1-\sigma F_z^*}{\rho-(1+\sigma-\sigma F_\pi^*)} & 1 \end{pmatrix} \begin{pmatrix} \rho^t & 0 \\ 0 & (1+\sigma-\sigma F_\pi^*)^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1-\sigma F_z^*}{\rho-(1+\sigma-\sigma F_\pi^*)} & 1 \end{pmatrix} \begin{pmatrix} z_0 \\ \pi_0 \end{pmatrix} \tag{61}$$

$$\begin{pmatrix} E_t z_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-\sigma F_z^*}{\rho-(1+\sigma-\sigma F_\pi^*)} \end{pmatrix} \rho^t z_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1+\sigma-\sigma F_\pi^*)^t \left(-\frac{1-\sigma F_z^*}{\rho-(1+\sigma-\sigma F_\pi^*)} z_0 + \pi_0 \right) \tag{62}$$