A DIAGNOSTIC CRITERION
FOR APPROXIMATE FACTOR STRUCTURE

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Abstract

We build a simple diagnostic criterion for approximate factor structure in large cross-sectional equity datasets. Given a model for asset returns with observable factors, the criterion checks whether the error terms are weakly cross-sectionally correlated or share at least one unobservable common factor. It only requires computing the largest eigenvalue of the empirical cross-sectional covariance matrix of the residuals of a large unbalanced panel. The panel data model accomodates both time-invariant and time-varying factor structures. We develop the theory for large cross-section and time-series dimensions. No restriction is imposed on the relation between both dimensions. The empirical analysis runs on returns for about ten thousands US stocks from July 1964 to December 2012. Among several multi-factor models proposed in the literature, we cannot select a model with zero factors in the errors.

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1 Introduction

Empirical work in asset pricing vastly relies on linear multi-factor models with either time-invariant coefficients (unconditional models) or time-varying coefficients (conditional models). The factor structure is often based on observable variables (empirical factors) and supposed to be rich enough to extract systematic risks while indiosyncratic risk is left over to the error term. Linear factor models are rooted in the Arbitrage Pricing Theory (APT, Ross (1976), Chamberlain and Rothschild (1983)) or come from a loglinearization of nonlinear consumption-based models (Campbell (1993)). Conditional linear factor models aim at capturing the time-varying influence of financial and macroeconomic variables in a simple setting (see e.g. Shanken (1990), Cochrane (1996), Ferson and Schadt (1996), Ferson and Harvey (1991, 1999), Lettau and Ludvigson (2001), Petkova and Zhang (2005)). Time variation in risk biases time-invariant estimates of alphas and betas, and therefore asset pricing test conclusions (Jagannathan and Wang (1996), Lewellen and Nagel (2006), Boguth et al. (2011)). Ghysels (1998) discusses the pros and cons of modeling time-varying betas.

A central and practical issue is to determine whether there are one or more factors omitted in the chosen specification. Approximate factor structures with nondiagonal error covariance matrices (Chamberlain and Rothschild (1983)) answer the potential empirical mismatch of exact factor structures with diagonal error covariance matrices underlying the original APT of Ross (1976). If the set of observable factors is correctly specified, the errors are weakly cross-sectionally correlated. Given the large menu of factors available in the literature (the factor zoo of Cochrane (2011), see also Harvey, Liu and Zhu (2013)), we need a simple diagnostic criterion to decide whether we can feel comfortable with the chosen set of observable factors.

For models with unobservable (latent) factors, Connor and Korajczyk (1993) were the first to develop a test for the number of factors for large balanced panels of individual stock returns in time-invariant models under covariance stationarity and homoskedasticity. Unobservable factors are estimated by the method of asymptotic principal components developed by Connor and Korajczyk (1986) (see also Stock and Watson (2002)). For heteroskedastic settings, the recent literature on large panels with static factors has extended the toolkit available to researchers. Bai and Ng (2002a) introduce a penalized least-squares strategy to estimate the number of factors, at least one, without restrictions on the relation between the cross-sectional dimension (n) and the time-series dimension (T). Caner and Han (forthcoming, 2014) propose an estimator with a group bridge penalization to determine the number of unobservable factors. Onatski (2009, 2010) looks at the behavior of the adjacent eigenvalues to determine the number of factors when n and T are comparable. Ahn and Horenstein (2013) opt for the same strategy and cover the possibility of zero factors. Kapetanios (2010) uses subsampling to estimate the limit distribution of the adjacent eigenvalues. In the spirit of Lehmann and Modest (1988) and Connor and Korajczyk (1988), Bai and Ng (2006) analyse statistics to test whether the observable factors in time-invariant models span the space of unobservable factors. They do not impose any restriction on n and T. They find that the three factor model of Fama and French (1993, FF) is the most satisfactory proxy for the unobservable factors estimated from balanced panels of portfolio and individual stock returns. Ahn, Horenstein and Wang (2013) study a rank estimation method to also check whether time-invariant factor models are compatible with a number of unobservable factors. For portfolio returns, they find that the FF model exhibits a full rank beta (factor loading) matrix.

In this paper, we build a simple diagnostic criterion for approximate factor structure in large cross-sectional datasets. The criterion checks whether the error terms in a given model with observable factors are weakly cross-sectionally correlated or share at least one common factor. It only requires computing the largest eigenvalue of the empirical cross-sectional covariance matrix of the residuals of a large unbalanced panel and substracting a penalization term vanishing to zero for large n and T. The steps of the diagnostic are easy: 1) compute the largest eigenvalue, 2) substract a penalty, 3) conclude to validity of the proposed approximate factor structure if the difference is negative, or conclude to at least one omitted factor if the difference is positive. Our theoretical contribution shows that step 3) yields asymptotically the correct model selection. We derive all properties for unbalanced panels in the setting of Connor and Korajczyk (1987) to avoid the survivorship bias inherent to studies restricted to balanced subsets of available stock return databases (Brown, Goetzmann, and Ross (1995)). The panel data model is sufficiently general to accomodate both time-invariant and time-varying factor structures (Gagliardini, Ossola, and Scaillet (2011), GOS). We develop the theory for large cross-section and time-series dimensions. No restriction is imposed on the relation between both dimensions. As shown below, the criterion is related to the penalized least-squares approach of Bai and Ng (2002a) for model selection with unobservable factors.

For our empirical contribution, we consider the Center for Research in Security Prices (CRSP) database and take the Compustat database to match firm characteristics. The merged dataset comprises about ten thousands stocks with monthly returns from July 1964 to December 2012. We look at factor models popular in the empirical finance literature to explain monthly equity returns. They differ by the choice of the factors. The first model is the CAPM (Sharpe (1964), Lintner (1965)) using the excess market return as the single factor. Then, we consider the three-factor model of Fama and French (1993) based on two additional factors capturing the book-to-market and size effects, and a four-factor extension including a momentum factor (Jegadeesh and Titman (1993), Carhart (1997)). We study time-invariant and time-varying versions of the factor models (Shanken (1990), Cochrane (1996), Ferson and Schadt (1996), Ferson and Harvey (1999)). For the latter, we use both macrovariables and firm characteristics as instruments (Avramov and Chordia (2006)). Among those multi-factor models, we cannot select a model with zero factors in the errors.

The outline of the paper is as follows. In Section 2, we consider a general framework of conditional linear factor model for asset returns. In Section 3, we present our diagnostic criterion for approximate factor structure. Section 5 contains the empirical results. We place all omitted proofs and the Monte Carlo simulation results in the online supplementary materials. There, we also include some additional empirical results and robustness checks.

2 Conditional factor model of asset returns

In this section, we consider a conditional linear factor model with time-varying coefficients. We work in a multi-period economy (Hansen and Richard (1987)) under an approximate factor structure (Chamberlain and Rothschild (1983)) with a continuum of assets as in GOS. Such a construction is close to the setting advocated by Al-Najjar (1995, 1998, 1999a) in a static framework with an exact factor structure. He discusses several key advantages of using a continuum economy in arbitrage pricing and risk decomposition. A key advantage is robustness of factor structures to asset repackaging (Al-Najjar (1999b); see GOS for a proof).

Let $\mathcal{F}_t$, with $t = 1, 2, ...$, be the information available to investors. Without loss of generality, the continuum of assets is represented by the interval $[0, 1]$. The excess returns $R_t(\gamma)$ of asset $\gamma \in [0, 1]$ at dates $t = 1, 2, ...$ satisfy the conditional linear factor model:

$$R_t(\gamma) = a_t(\gamma) + b_t(\gamma)' f_t + \varepsilon_t(\gamma), \tag{1}$$

where vector f_t gathers the values of K observable factors at date t. The intercept $a_t(\gamma)$ and factor sensitivities $b_t(\gamma)$ are $\mathcal{F}_{t-1}$-measurable. The error terms $\varepsilon_t(\gamma)$ have mean zero and are uncorrelated with the factors conditionally on information $\mathcal{F}_{t-1}$. Moreover, we exclude asymptotic arbitrage opportunities in the economy: there are no portfolios that approximate arbitrage opportunities when the number of assets increases. In this setting, GOS show that the following asset pricing restriction holds:

$$a_t(\gamma) = b_t(\gamma)'\nu_t, \text{ for almost all } \gamma \in [0,1], \tag{2}$$

almost surely in probability, where random vector $\nu_t \in \mathbb{R}^K$ is unique and is $\mathcal{F}_{t-1}$-measurable. The asset pricing restriction (2) is equivalent to $E[R_t(\gamma)|\mathcal{F}_{t-1}] = b_t(\gamma)'\lambda_t$, where $\lambda_t = \nu_t + E[f_t|\mathcal{F}_{t-1}]$ is the vector of the conditional risk premia.

To have a workable version of Equations (1) and (2), we define how the conditioning information is generated and how the model coefficients depend on it via simple functional specifications. The conditioning information $\mathcal{F}_{t-1}$ contains $Z_{\underline{t-1}}$ and $Z_{\underline{t-1}}(\gamma)$, for all $\gamma \in [0,1]$, where the vector of lagged instruments $Z_{t-1} \in \mathbb{R}^p$ is common to all stocks, the vector of lagged instruments $Z_{t-1}(\gamma) \in \mathbb{R}^q$ is specific to stock γ, and $Z_{\underline{t}} = \{Z_t, Z_{t-1}, ...\}$. Vector Z_{t-1} may include the constant and past observations of the factors and some additional variables such as macroeconomic variables. Vector $Z_{t-1}(\gamma)$ may include past observations of firm characteristics and stock returns. To end up with a linear regression model, we assume that: (i) the vector of factor loadings $b_t(\gamma)$ is a linear fuction of lagged instruments Z_{t-1} (Shanken (1990), Ferson and Harvey (1991)) and $Z_{t-1}(\gamma)$ (Avramov and Chordia (2006)); (ii) the vector of risk premia λ_t is a linear function of lagged instruments Z_{t-1} (Cochrane (1996), Jagannathan and Wang (1996)); (iii) the conditional expectation of f_t given the information $\mathcal{F}_{t-1}$ depends on Z_{t-1} only and is linear (as e.g. if Z_t follows a Vector Autoregressive (VAR) model of order 1).

To ensure that cross-sectional limits exist and are invariant to reordering of the assets, we introduce a sampling scheme as in GOS. We formalize it so that observable assets are random draws from an underlying population (Andrews (2005)). In particular, we rely on a sample of n assets by randomly drawing i.i.d. indices γ_i from the population according to a probability distribution G on $[0,1]$. For any $n, T \in \mathbb{N}$, the excess returns are $R_{i,t} = R_t(\gamma_i)$. Similarly, let $a_{i,t} = a_t(\gamma_i)$ and $b_{i,t} = b_t(\gamma_i)$ be the characteristics, and $\varepsilon_{i,t} = \varepsilon_t(\gamma_i)$ be the error terms. By random sampling, we get a random coefficient panel model (e.g. Hsiao (2003), Chapter 6). In available datasets, we do not observe asset returns for all firms at all dates. Thus, we account for the unbalanced nature of the panel through a collection of indicator variables $I_{i,t}$, for any asset i at time t. We define $I_{i,t} = 1$ if the return of asset i is observable at date t, and 0 otherwise (Connor and

Korajczyk (1987)).

Through appropriate redefinitions of the regressors and coefficients, GOS show that we can rewrite the model for Equations (1) and (2) as follows:

$$R_{i,t} = x'_{i,t}\beta_i + \varepsilon_{i,t}, \tag{3}$$

where the regressor $x_{i,t} = \left(x'_{1,i,t}, x'_{2,i,t}\right)'$ has dimension $d = d_1 + d_2$ and includes vectors $x_{1,i,t} = \left(vech\left[X_t\right]', Z'_{t-1} \otimes Z'_{i,t-1}\right)' \in \mathbb{R}^{d_1}$ and $x_{2,i,t} = \left(f'_t \otimes Z'_{t-1}, f'_t \otimes Z'_{i,t-1}\right)' \in \mathbb{R}^{d_2}$ with $d_1 = p(p+1)/2 + pq$ and $d_2 = K(p+q)$. The symmetric matrix $X_t = [X_{t,k,l}] \in \mathbb{R}^{p \times p}$ is such that $X_{t,k,l} = Z^2_{t-1,k}$, if $k = l$, and $X_{t,k,l} = 2Z_{t-1,k}Z_{t-1,l}$, otherwise, $k, l = 1, \ldots, p$. The vector-half operator $vech\left[\cdot\right]$ stacks the elements of the lower triangular part of a $p \times p$ matrix as a $p\left(p+1\right)/2 \times 1$ vector (see Chapter 2 in Magnus and Neudecker (2007) for properties of this matrix tool). In matrix notation, for any asset i, we have

$$R_i = X_i\beta_i + \varepsilon_i, \tag{4}$$

where R_i and ε_i are $T \times 1$ vectors. Regression (3) contains both explanatory variables that are common across assets (scaled factors) and asset-specific regressors. It includes models with time-invariant coefficients as a particular case. In such a case, the regressor reduces to $x_t = (1, f'_t)'$ and is common across assets.

In order to build the diagnostic criterion for the set of observable factors, we consider the following rival models:

$\mathcal{M}_1$: the linear regression model (3), where the errors $(\varepsilon_{i,t})$ follow an approximate factor structure,

and

$\mathcal{M}_2$: the linear regression model (3), where the errors $(\varepsilon_{i,t})$ satisfy a factor structure.

Thus, the error terms $\varepsilon_{i,t}$ are weakly cross-sectionally dependent under model $\mathcal{M}_1$. On the other hand, under model $\mathcal{M}_2$, the following error factor structure holds

$$\varepsilon_{i,t} = \theta'_i h_t + u_{i,t}, \tag{5}$$

where the $m \times 1$ vector h_t includes unobservable (i.e., latent or hidden) factors. The $m \times 1$ vector θ_i corresponds to the factor loadings, and the number m of common factors is assumed unknown. In vector notation, we have:

$$\varepsilon_i = H\theta_i + u_i, \tag{6}$$

where H is the $T \times m$ matrix of unobservable factor values, and u_i is a $T \times 1$ vector.

Model $\mathcal{M}_2$ can be distinguished from model $\mathcal{M}_1$ only if the systematic component $H\theta_i$ in the error vector ε_i is not spanned by the columns of matrix X_i for most assets. Otherwise, the common component $H\theta_i$ can be absorbed in the observable regressors, and we face an identification issue. Therefore, we introduce the next assumption under model $\mathcal{M}_2$.

Assumption 1 *Under model $\mathcal{M}_2$, the factor structure in the error term is such that*

$$\mu_1\left(\frac{1}{nT}\sum_i M_{\tilde{X}_i}\tilde{H}_i\theta_i\theta_i'\tilde{H}_i'M_{\tilde{X}_i}\right) \geq c, \text{ with probability approaching } 1,$$

for a constant $c > 0$, where $M_{\tilde{X}_i} = I_T - \tilde{X}_i\left(\tilde{X}_i'\tilde{X}_i\right)^{-1}\tilde{X}_i'$ for any i, I_T denotes the identity matrix of dimension T, $\tilde{X}_i = \boldsymbol{I}_i \odot X_i$, $\tilde{H}_i = \boldsymbol{I}_i \odot H$, with $\boldsymbol{I}_i$ the $T \times 1$ vector of observability indicators of asset i, and $\mu_1(\cdot)$ and $\odot$ denote the largest eigenvalue of a symmetric matrix and the Hadamard product between matrices, respectively.

In Assumption 1, vector $\hat{\eta}_i = M_{\tilde{X}_i}\tilde{H}_i\theta_i$ is the residual vector in the regression of $\tilde{H}_i\theta_i$ on the explanatory variables $\tilde{X}_i$. Thus, $\hat{\eta}_i$ is the part of the systematic component of the error vector, which is not spanned by the observable regressors. Assumption 1 requires that the largest eigenvalue of the cross-sectional second-order moments matrix of vectors $\hat{\eta}_i$, standardized by their dimension T, does not vanish asymptotically. By the literature on unobservable factor models (e.g., Bai and Ng (2002a)) this condition amounts to the presence of some common factos in the $\hat{\eta}_i$. For a factor model with time invariant coefficients and a balanced panel, we have $\tilde{X}_i = X_i = X$, and $\tilde{H}_i = H$ (see Appendix A.2.1):

$$\mu_1\left(\frac{1}{nT}\sum_i M_X H\theta_i\theta_i'H'M_X\right) \geq \mu_m\left(\frac{\Theta'\Theta}{n}\right)\mu_1\left(\frac{H'M_XH}{T}\right), \tag{7}$$

where Θ is the $n \times m$ matrix of factor loadings and $\mu_m(\cdot)$ is the smallest eigenvalue of a $m \times m$ symmetric matrix. Without loss of generality, the unobservable factors can be selected orthogonal to the observable regressors. Thus, Assumption 1 is satisfied, if matrices $\frac{\Theta'\Theta}{n}$ and $\frac{H'H}{T}$, i.e., the second-moments matrices of the loadings and of the factors converge to positive definite matrices (see Assumptions A and B in Bai and Ng (2002a)).

3 Diagnostic criterion

In this section, we provide the diagnostic criterion that checks whether the error terms are weakly cross-sectionally correlated or share at least one common factor. To compute the criterion, we estimate model (3) by OLS asset by asset, and we get estimators $\hat{\beta}_i = \hat{Q}_{x,i}^{-1} \frac{1}{T_i} \sum_t I_{i,t} x_{i,t} R_{i,t}$, for $i = 1, ..., n$, where $\hat{Q}_{x,i} = \frac{1}{T_i} \sum_t I_{i,t} x_{i,t} x_{i,t}'$. We get the residuals $\hat{\varepsilon}_{i,t} = R_{i,t} - x_{i,t}' \hat{\beta}_i$, where $\hat{\varepsilon}_{i,t}$ is observable only if $I_{i,t} = 1$. In available panels, the random sample size T_i for asset i can be small, and the inversion of matrix $\hat{Q}_{x,i}$ can be numerically unstable. To avoid unreliable estimates of β_i, we apply a trimming approach as in GOS. We define $\mathbf{1}_i^\chi = \mathbf{1}\left\{CN\left(\hat{Q}_{x,i}\right) \leq \chi_{1,T}, \tau_{i,T} \leq \chi_{2,T}\right\}$, where $CN\left(\hat{Q}_{x,i}\right) = \sqrt{\mu_1\left(\hat{Q}_{x,i}\right) / \mu_d\left(\hat{Q}_{x,i}\right)}$ is the condition number of matrix $\hat{Q}_{x,i}$, and $\tau_{i,T} = T/T_i$. The two sequences $\chi_{1,T} > 0$ and $\chi_{2,T} > 0$ diverge asymptotically. The first trimming condition $\{CN\left(\hat{Q}_{x,i}\right) \leq \chi_{1,T}\}$ keeps in the cross-section only assets for which the time series regression is not too badly conditioned. A too large value of $CN\left(\hat{Q}_{x,i}\right)$ indicates multicollinearity problems and ill-conditioning (Belsley, Kuh, and Welsch (2004), Greene (2008)). The second trimming condition $\{\tau_{i,T} \leq \chi_{2,T}\}$ keeps in the cross-section only assets for which the time series is not too short. We also use both trimming conditions in the proofs of the asymptotic results.

We consider the following diagnostic criterion:

$$\xi = \mu_1\left(\frac{1}{nT} \sum_i \mathbf{1}_i^\chi \bar{\varepsilon}_i \bar{\varepsilon}_i'\right) - g(n, T), \tag{8}$$

where the vector $\bar{\varepsilon}_i$ of dimension T gathers the values $\bar{\varepsilon}_{i,t} = I_{i,t} \hat{\varepsilon}_{i,t}$, the penalty $g(n, T)$ is such that $g(n, T) \to 0$ and $C_{n,T}^2 g(n, T) \to \infty$, when $n, T \to \infty$, for $C_{n,T}^2 = \min\{n, T\}$. Bai and Ng (2002a) consider several simple potential candidates for the penalty $g(n, T)$. We list and implement them in Section 4. In vector $\bar{\varepsilon}_i$, the unavailable residuals are replaced by zeros. The following model selection rule explains our choice of the diagnostic criterion (8) for approximate factor structure in large unbalanced cross-sectional datasets.

Proposition 1 *Model selection rule: Under Assumption 1 and Assumptions A.1-A.5, (a) we select* $\mathcal{M}_1$ *if* $\xi < 0$*, since* $Pr\left(\xi < 0 \mid \mathcal{M}_1\right) \to 1$*, when* $n, T \to \infty$*; (b) we select* $\mathcal{M}_2$ *if* $\xi > 0$*, since* $Pr\left(\xi > 0 \mid \mathcal{M}_2\right) \to 1$*, when* $n, T \to \infty$*.*

Proposition 1 characterizes an asymptotically valid model selection rule, which treats both models symmetrically. This is not a testing procedure since we do not use a critical region based an asymptotic distribution and a chosen significance level. The proof of Proposition 1 shows that the largest eigenvalue in (8) vanishes at a faster rate than the penalization term under $\mathcal{M}_1$ when n and T go to infinity. This explains why we select the first model when ξ is negative. On the contrary, the largest eigenvalue remains bounded from below away from zero under $\mathcal{M}_2$ when n and T go to infinity. This explains why we select the second model when ξ is positive. The criterion (8) can be interpreted as the adjusted gain in fit including a single additional (unobservable) factor in model $\mathcal{M}_1$. In the balanced case, where $I_{i,t} = 1$ for all i and t, we can rewrite (8) as $\xi = SS_0 - SS_1 - g(n,T)$, where $SS_0 = \frac{1}{nT}\sum_i \sum_t \hat{\varepsilon}_{i,t}^2$ is the sum of squared errors and $SS_1 = \min \frac{1}{nT}\sum_i \sum_t (\hat{\varepsilon}_{i,t} - \theta_i h_t)^2$, where the minimization is w.r.t. the vectors $H \in \mathbb{R}^T$ of factor values and $\Theta \in \mathbb{R}^n$ of factor loadings in a one-factor model, subject to the normalization constraint $\frac{H'H}{T} = 1$. Indeed, the largest eigenvalue $\mu_1 \left(\frac{1}{nT}\sum_i \hat{\varepsilon}_i \hat{\varepsilon}_i' \right)$ corresponds to the difference between SS_0 and SS_1. Furthermore, the criterion ξ is equal to the difference of the penalized criteria for zero- and one-factor models defined in Bai and Ng (2002a) applied on the residuals. Indeed, $\xi = PC(0) - PC(1)$, where $PC(0) = SS_0$, and $PC(1) = SS_1 + g(n,T)$. Given such an interpretation in terms of sums of squared errors, we can suggest another diagnostic criterion based on a logarithmic transform as in Corollary 2 of Bai and Ng (2002a). The second diagnostic criterion is

$$\check{\xi} = \ln\left(\frac{1}{nT}\sum_i \sum_t \mathbf{1}_i^{\chi} \hat{\varepsilon}_{i,t}^2 \right) - \ln\left(\frac{1}{nT}\sum_i \sum_t \mathbf{1}_i^{\chi} \hat{\varepsilon}_{i,t}^2 - \mu_1\left(\frac{1}{nT}\sum_i \mathbf{1}_i^{\chi} \hat{\varepsilon}_i \hat{\varepsilon}_i' \right) \right) - g(n,T). \quad (9)$$

In the balanced case, we get $\check{\xi} = \ln(SS_0/SS_1) - g(n,T)$ and it is equal to the difference of $IC(0)$ and $IC(1)$ criteria in Bai and Ng (2002a). The following proposition states the model selection rule based on $\check{\xi}$.

Proposition 2 *The model selection rule is the same as in Proposition 1 with $\check{\xi}$ substituted for ξ.*

The recent literature on the properties of the two-pass regressions for fixed n and large T shows that the presence of useless factors (Kan and Zhang (1999a,b), Gospodinov, Kan and Robotti (2014)) or weak factor loadings (Kleibergen (2009)) does not affect the asymptotic distributional properties of factor loading estimates, but alters the ones of the risk premia estimates. Useless factors have zero loadings, and weak loadings drift to zero at rate $1/\sqrt{T}$. The vanishing rate of the largest eigenvalue of the empirical cross-sectional covariance matrix of the residuals does not change if we face useless factors or weak factor loadings in the observable factors under $\mathcal{M}_1$. The same remark applies under $\mathcal{M}_2$. Hence the selection rule remains the same since the probability of taking the right decision still approaches 1. If we have a number of useless factors or weak factor loadings strictly lower than the number m of the omitted factors under $\mathcal{M}_2$, this does not impact the asymptotic rate of the diagnostic criterion if Assumption 1 holds. If we only have useless factors in the omitted factors under $\mathcal{M}_2$, we face an identification issue. Assumption 1 is not satisfied. We cannot distinguish such a specification from $\mathcal{M}_1$ since it corresponds to a particular approximate factor structure. Again the selection rule remains the same since the probability of taking the right decision still approaches 1. Finally, let us study the case of only weak factor loadings under $\mathcal{M}_2$. We consider a simplified setting:

$$R_{i,t} = x_{i,t}'\beta_i + \epsilon_{i,t}$$

where $\epsilon_{i,t} = \theta_i h_t + u_{i,t}$ has only one factor with a weak factor loading, namely $m = 1$ and $\theta_i = \bar{\theta}_i / T^{\gamma}$ with $\gamma > 0$. Let us assume that $\mu_1 \left(\frac{1}{nT} \sum_i M_{\tilde{X}_i} \tilde{H}_i \tilde{H}_i' M_{\tilde{X}_i} \bar{\theta}_i^2 \right)$ is bounded from below away from zero (see Assumption 1) and bounded from above. By the properties of the eigenvalues of a scalar multiple of a matrix, we deduce that $c_1/T^{2\gamma} \leq \mu_1 \left(\frac{1}{nT} \sum_i M_{\tilde{X}_i} \tilde{H}_i \tilde{H}_i' M_{\tilde{X}_i} \theta_i^2 \right) \leq c_2/T^{2\gamma}$, for some constants c_1, c_2 such that $c_2 \geq c_1 > 0$. Hence, by similar arguments as in the proof of Proposition 1, we get:

$$c_1 T^{-2\gamma} - g(n,T) + O_p\left(C_{nT}^{-2} + \bar{\chi}_T T^{-1}\right) \leq \xi \leq c_2 T^{-2\gamma} - g(n,T) + O_p\left(C_{nT}^{-2} + \bar{\chi}_T T^{-1}\right),$$

where we define $\bar{\chi}_T = \chi_{1,T}^4 \chi_{2,T}^2$. To conclude $\mathcal{M}_2$, we need that $C_{nT}^{-2} + \bar{\chi}_T T^{-1}$ and the penalty $g(n,T)$ vanish at a faster rate than $T^{-2\gamma}$, namely $C_{nT}^{-2} + \bar{\chi}_T T^{-1} = o\left(T^{-2\gamma}\right)$ and $g(n,T) = o\left(T^{-2\gamma}\right)$. To conclude $\mathcal{M}_1$, we need that $g(n,T)$ is the dominant term, namely $T^{-2\gamma} = o\left(g(n,T)\right)$ and $C_{nT}^{-2} + \bar{\chi}_T T^{-1} = o\left(g(n,T)\right)$. As an example, let us take $g(n,T) = T^{-1}\log T$ and $n = T^{\bar{\gamma}}$ with $\bar{\gamma} > 1$, and assume that the trimming is such that $\bar{\chi}_T = o(\log T)$. Then, we conclude $\mathcal{M}_2$ if $\gamma < 1/2$ and $\mathcal{M}_1$ if $\gamma > 1/2$. This means that detecting a weak factor loading structure is difficult if gamma is not sufficiently small. The factor loading should drift to zero not too fast to conclude $\mathcal{M}_2$. Otherwise, we cannot distinguish it asymptotically from an approximate factor structure.

4 Determining the number of factors

In the previous section, we have studied a diagnostic criterion to check whether the error terms are weakly cross-sectionally correlated or share at least one unobservable common factor. This section aims at answering: do we have one, two, or more omitted factors? The design of the diagnostic criterion to check whether the error terms share exactly k unobservable common factors or share at least $k+1$ unobservable common factors follows the same mechanics. We consider the following rival models:

$\mathcal{M}_1(k)$: the linear regression model (3), where the errors $(\varepsilon_{i,t})$ satisfy a factor structure with exactly k unobservable factors,

and

$\mathcal{M}_2(k)$: the linear regression model (3), where the errors $(\varepsilon_{i,t})$ satisfy a factor structure with at least $k+1$ unobservable factors.

The following assumption consists of identification assumptions similar to Assumption 1, but for exactly k unobservable factors and at least $k+1$ unobservable factors in the error terms.

Assumption 2 *a) Under model $\mathcal{M}_1(k)$, the factor structure in the error term is such that*

$$\mu_k\left(\frac{1}{nT}\sum_i M_{\tilde{X}_i}\tilde{H}_i\theta_i\theta_i'\tilde{H}_i'M_{\tilde{X}_i}\right)\geq c,\quad \textit{with probability approaching 1},$$

for a constant $c>0$. b) Under model $\mathcal{M}_2(k)$, the factor structure in the error term is such that

$$\mu_{k+1}\left(\frac{1}{nT}\sum_i M_{\tilde{X}_i}\tilde{H}_i\theta_i\theta_i'\tilde{H}_i'M_{\tilde{X}_i}\right)\geq c,\quad \textit{with probability approaching 1},$$

for a constant $c>0$.

For a factor model with time invariant coefficients and a balanced panel, Assumption 2 is satisfied if matrices $\dfrac{\Theta'\Theta}{n}$ and $\dfrac{H'M_XH}{T}$ converge to positive definite matrices since we deduce as in Section 2 from the inequalities in Wang and Zhang (1992):

$$\mu_k\left(\frac{1}{nT}\sum_1 M_XH\theta_i\theta_i'H'M_X\right)\geq\mu_m\left(\frac{\Theta'\Theta}{n}\right)\mu_k\left(\frac{H'M_XH}{T}\right).$$

The diagnostic criterion exploits the kth largest eigenvalue of the empirical cross-sectional covariance matrix of the residuals:

$$\xi(k) = \mu_{k+1}\left(\frac{1}{nT}\sum_i \mathbf{1}_i^{\chi}\bar{\varepsilon}_i\bar{\varepsilon}_i'\right) - g(n,T). \tag{10}$$

As discussed in Ahn and Horenstein (2013) (see also Onatski (2013)), we can rewrite (10) in the balanced case as $\xi(k) = SS_k - SS_{k+1} - g(n,T)$ where SS_k equals the sample mean of the squared residuals from the time series regressions of individual response variables $(\hat{\varepsilon}_{i,t})$ on the first k principal components of $\frac{1}{nT}\sum_i \hat{\varepsilon}_i\hat{\varepsilon}_i'$. The criterion $\xi(k)$ is equal to the difference of the penalized criteria for k and $(k+1)$- factor models defined in Bai and Ng (2002a) applied on the residuals. Indeed, $\xi(k) = PC(k) - PC(k+1)$, where $PC(k) = SS_k + kg(n,T)$, and $PC(k+1) = SS_{k+1} + (k+1)g(n,T)$. To determine the number of unobservable factors, we choose the minimum k such that $\xi(k) < 0$. Graphically, we can build a penalised scree plot where we display the penalised eigenvalues associated with each factor in descending order versus the number of the factor, and use the x-axis for the cut-off point. The following model selection rule extends Proposition 1 to determine the number of factors.

Proposition 3 *Model selection rule: under Assumptions..., (a) we select $\mathcal{M}_1(k)$ if $\xi(k) < 0$, since $Pr[\xi(k) < 0|\mathcal{M}_1(k)] \to 1$, when $n, T \to \infty$ such that $n = O(T^2)$; (b) we select $\mathcal{M}_2(k)$ if $\xi(k) > 0$, since $Pr[\xi(k) > 0|\mathcal{M}_2(k)] \to 1$, when $n, T \to \infty$.*

In Proposition 3 part a), we need the additional constraint $n = O(T^2)$ on the relative rate of the cross-sectional dimension w.r.t. the time series dimension. The contribution $\mu_{k+1}\left(\frac{1}{nT}\sum_i \tilde{H}_i\theta_i\theta_i'\tilde{H}_i'\right) = O_p(1/\sqrt{\max\{n,T\}})$ coming from the k omitted factors (Lemma ... in the appendix) does not dominate asymptotically the contribution $\mu_1\left(\frac{1}{nT}\sum_i \tilde{u}_i\tilde{u}_i'\right) = C_{n,T}^{-2}$ under $\mathcal{M}_1(k)$ when $n = O(T^2)$. We do not need such an additional restriction in a balanced panel since $\mu_{k+1}\left(H\left(\frac{1}{nT}\sum_i \theta_i\theta_i'\right)H'\right) = 0$ if we have exactly k factors. This exemplifies a key difference between the asymptotics for balanced and unbalanced panels, and the proportional asymptotics used in Onatski (2009, 2010) or Ahn and Horenstein (2013). Those papers rely on the asymptotic distribution of the eigenvalues of large dimensional sample covariances matrices when $n/T(n) \to c > 0$ as $n \to \infty$. The condition $n = O(T^2)$ agrees with the "large n, small T" case that we face in the empirical application (ten thousands individual stocks monitored over forty-five years of monthly returns). The proof of Proposition 3 is also more complicated than the proof of Proposition 1. The proof of the latter directly exploits the equality between the largest value of a symmetric matrix and its operator norm, the triangular inequality of the matrix norm, and its upper bound given by the Frobenius norm. We need additional arguments based on Weyl inequalities (Theorem 4.3.1 in Horn and Johnson (1985)) when we look at the $k + 1$th eigenvalue.

5 Empirical results

5.1 Factor models and data description

We consider a vector of four factors $f_t = (r_{m,t}, r_{smb,t}, r_{hml,t}, r_{mom,t})'$, where $r_{m,t}$ is the month t excess return on CRSP NYSE/AMEX/Nasdaq value-weighted market portfolio over the risk free rate, and $r_{smb,t}$, $r_{hml,t}$ and $r_{mom,t}$ are the month t returns on zero-investment factor-mimicking portfolios for size, book-to-market, and momentum (see Fama and French (1993), Jegadeesh and Titman (1993), Carhart (1997)). We have downloaded the time series of factors from the website of Kenneth French. We proxy the risk free rate with the monthly 30-day T-bill beginning-of-month yield. To account for time-varying coefficients, we use a conditional specification based on two common variables and a firm-level variable. We take the instruments $Z_t = (1, \, Z_t^{*\prime})'$, where bivariate vector Z_t^* includes the term spread, proxied by the difference between yields on 10-year Treasury and 3-month T-bill, and the default spread, proxied by the yield difference between Moody's Baa-rated and Aaa-rated corporate bonds. We take a scalar $Z_{i,t}$ corresponding to the book-to-market equity of firm i. We refer to Avramov and Chordia (2006) for convincing theoretical and empirical arguments in favor of the chosen conditional specification. The vector $x_{i,t}$ has dimension $d = 25$, and parsimony explains why we have not included e.g. the size of firm i as an additional stock specific instrument.

We compute the firm characteristics from Compustat as in the appendix of Fama and French (2008). The CRSP database provides the monthly stock returns data and we exclude financial firms (Standard Industrial Classification Codes between 6000 and 6999) as in Fama and French (2008). The dataset after matching CRSP and Compustat contents comprises $n = 9,936$ stocks, and covers the period from July 1964 to

December 2009 with $T = 546$ months.

5.2 Diagnostic results

In this section, we compute the diagnostic criterion (8) assuming time-invariant and time-varying specifications of the linear factor model (1). We first define the specification for the penalty $g(n,T)$ in (8). Bai and Ng (2002a) propose three choices for the penalty function, leading to the following criteria:

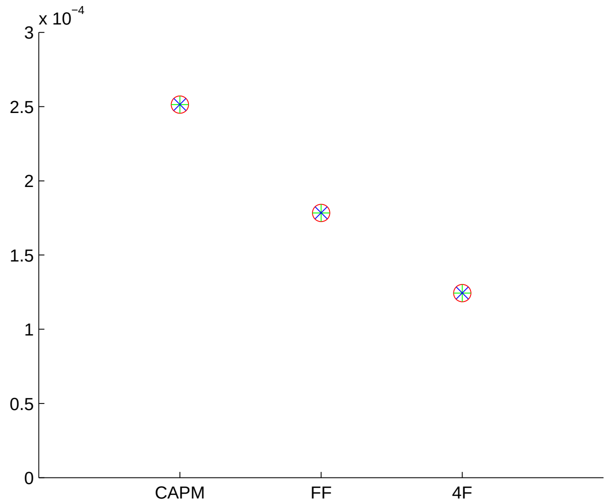
1. $\xi_1 = \mu_1 \left(\frac{1}{nT} \sum_i \mathbf{1}_i^{\chi} \bar{\varepsilon}_i \bar{\varepsilon}_i' \right) - \hat{\sigma}^2 \left(\frac{n+T}{nT} \right) \ln \left(\frac{nT}{n+T} \right)$;

2. $\xi_2 = \mu_1 \left(\frac{1}{nT} \sum_i \mathbf{1}_i^{\chi} \bar{\varepsilon}_i \bar{\varepsilon}_i' \right) - \hat{\sigma}^2 \left(\frac{n+T}{nT} \right) \ln C_{nT}^2$;

3. $\xi_3 = \mu_1 \left(\frac{1}{nT} \sum_i \mathbf{1}_i^{\chi} \bar{\varepsilon}_i \bar{\varepsilon}_i' \right) - \hat{\sigma}^2 \left(\frac{\ln C_{nT}^2}{C_{nT}^2} \right)$,

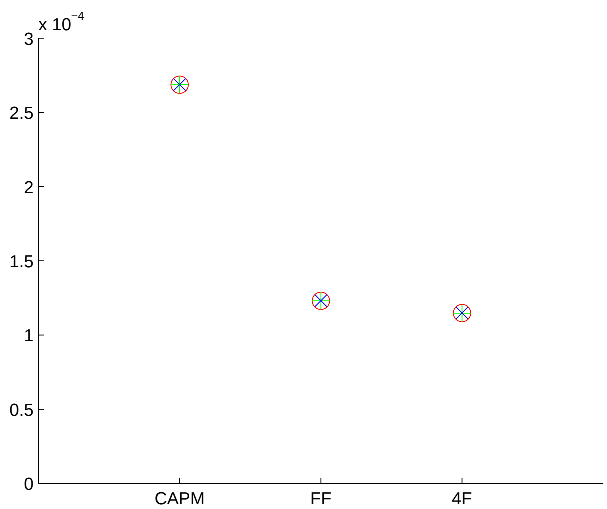
where $\hat{\sigma}^2 = \frac{1}{nT} \sum_i \sum_t \mathbf{1}_i^{\chi} \bar{\varepsilon}_{1,i,t}^2$, and $\bar{\varepsilon}_{1,i,t}$ are the fitted residuals of the four-factors model. In order to compute the diagnostic criteria, we estimate time-invariant and time-varying factor models. We fix $\chi_{1,T} = 15$ as advocated by Greene (2008), and $\chi_{2,T} = 546/12$ for the time-invariant estimation and $\chi_{1,T} = 20$ and $\chi_{2,T} = 546/60$ for the time-varying estimation. Figure 1 plots the values of the diagnostic criteria ξ_1, ξ_2 and ξ_3 for the time-invariant CAPM, FF and four-factors models. Figure 2 plots the values of the diagnostic criteria ξ_1, ξ_2 and ξ_3 for the time-varying CAPM, FF and four-factors models. The diagnostic criteria are positive. We cannot select a model with zero factors in the errors.

Figure 1: Estimated values of the diagnostic criteria ξ_1, ξ_2 and ξ_3 for the time-invariant models



The figure plots the values of the diagnostic criteria ξ_1(red circle), ξ_2 (green plus sign) and ξ_3 (blue cross) for the three time-invariant specifications of CAPM, FF, and four-factor (4F) models.

Figure 2: Estimated values of the diagnostic criteria ξ_1, ξ_2 and ξ_3 for the time-varying models



The figure plots the values of the diagnostic criteria ξ_1(red circle), ξ_2 (green plus sign) and ξ_3 (blue cross) for the three time-varying specifications of CAPM, FF, and four-factor (4F) models.

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Appendix 1: Regularity conditions

In this Appendix, we list and comment additional assumptions used to derive the proofs in Appendix 2.

Assumption A.1 *There exists a constant $M > 0$ such that, for all $n, T \in \mathbb{N}$, we have:*
a) $\frac{1}{n^2 T} \sum_{i,j} \sum_{t_1,t_2,t_3} \left| E\left[\varepsilon_{i,t_1}\varepsilon_{j,t_2}\varepsilon_{i,t_3}\varepsilon_{j,t_3} \left| x_{i,\underline{T}}, x_{j,\underline{T}}, \gamma_i, \gamma_j \right.\right]\right| \leq M;$
b) $\frac{1}{n^2 T^2} \sum_{i,j} \sum_{t_1,t_2,t_3,t_4} \left| E\left[\varepsilon_{i,t_1}\varepsilon_{i,t_2}\varepsilon_{j,t_3}\varepsilon_{j,t_4} \left| x_{i,\underline{T}}, x_{j,\underline{T}}, \gamma_i, \gamma_j \right.\right]\right| \leq M.$

Assumption A.2 *The error terms $(\varepsilon_{i,t})$ are $\varepsilon_{i,t} = u_{i,t}$ under model $\mathcal{M}_1$, and $\varepsilon_{i,t} = \theta_i' h_t + u_{i,t}$ under model $\mathcal{M}_2$, where the $(u_{i,t})$ are such that for a constant $M > 0$ and for all $n, T \in \mathbb{N}$ we have:*
a) $\frac{1}{nT} \sum_{i,j} \sum_{t} \left| E\left[u_{i,t}u_{j,t} \left| h_{\underline{T}}, \gamma_i, \gamma_j \right.\right]\right| \leq M;$
b) $\frac{1}{nT} \sum_{i,j} \sum_{t_1,t_2} \left| E\left[u_{i,t_1}u_{j,t_2} \left| h_{\underline{T}}, x_{i,\underline{T}}, x_{j,\underline{T}}, \gamma_i, \gamma_j \right.\right]\right| \leq M;$
c) $\frac{1}{n^2 T} \sum_{i,j} \sum_{t_1,t_2} \left| E\left[u_{i,t_1}u_{j,t_1}u_{i,t_2}u_{j,t_2}\right]\right| \leq M;$
d) $\frac{1}{n^2 T} \sum_{i,j} \sum_{t_1,t_2,t_3} \left| E\left[u_{i,t_1}u_{j,t_2}u_{i,t_3}u_{j,t_3} \left| h_{\underline{T}}, x_{i,\underline{T}}, x_{j,\underline{T}}, \gamma_i, \gamma_j \right.\right]\right| \leq M;$
e) $\frac{1}{n^2 T^2} \sum_{i,j} \sum_{t_1,t_2,t_3,t_4} \left| E\left[u_{i,t_1}u_{i,t_2}u_{j,t_3}u_{j,t_4} \left| h_{\underline{T}}, x_{i,\underline{T}}, x_{j,\underline{T}}, \gamma_i, \gamma_j \right.\right]\right| \leq M.$
f) $E\left[\varepsilon_{i,t}^4\right] \leq M$*, for all $i \leq n$ and $t \leq T$.*

Assumption A.3 *There exists a constant $M > 0$ such that $\|x_{i,t}\| \leq M$, P-a.s., for any i and t.*

Assumption A.4 *a) There exists a constant $M > 0$ such that $\|h_t\| \leq M$, P-a.s., for all t. Moreover, b) $\|\theta_i\| < M$, for all i.*

Assumption A.5 *The trimming constants $\chi_{1,T}$ and $\chi_{2,T}$ are such that $\chi_{1,T}^2 \chi_{2,T} = o\left(Tg\left(n,T\right)\right)$.*

Assumption A.6 *Under model $\mathcal{M}_1$, $\tilde{\mathcal{E}} = (\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_i, \ldots, \tilde{\varepsilon}_N) = E_{1T}^{1/2} E_2 E_{3N}^{1/2}$, where $\tilde{\varepsilon}_i = I_i \odot \varepsilon_i$, and under model $\mathcal{M}_2$, $\tilde{\mathcal{U}} = (\tilde{u}_1, \ldots, \tilde{u}_i, \ldots, \tilde{u}_N)$, where $\tilde{u}_i = I_i \odot u_i$, such that (i) $E_2' = (e_{it})$, and $E_{1T}^{1/2}$ and $E_{3N}^{1/2}$ are the symmetric square roots of $T \times T$ and $N \times N$ positive semidefinite matrices E_{1T} and E_{3N}, respectively, (ii) the e_{it} are independent and identically distributed (i.i.d.) random variables with zero mean* *and uniformly bounded moments up to the fourth order, (iii) $\mu_1(E_{1T})$ and $\mu_1(E_{3N})$ are bounded from above uniformly in T and N, respectively.*

Assumption A.6 is the same as in Ahn and Horenstein (2013) (see also Onatski (2010))). It allows for correlation and heteroskedasticity in both dimensions of $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{U}}$. Bai and Ng (2002b) show that those assumptions of weak cross-section and serial correlations ensure that $\mu_1\left(\frac{1}{nT}\sum_i \tilde{\varepsilon}_i\tilde{\varepsilon}_i'\right) = O_p\left(C_{nT}^{-2}\right)$ under $\mathcal{M}_1$, and $\mu_1\left(\frac{1}{nT}\sum_i \tilde{u}_i\tilde{u}_i'\right) = O_p\left(C_{nT}^{-2}\right)$ under $\mathcal{M}_2$.

Appendix 2: Proofs

A.2.1 Derivation of Equation (7)

We have:

$$\begin{aligned}
\mu_1\left(\frac{1}{nT}\sum_i M_X H\theta_i\theta_i' H' M_X\right) &= \mu_1\left(\frac{1}{T}M_X H\left(\frac{1}{n}\sum_i \theta_i\theta_i'\right)H'M_X\right) \\
&= \mu_1\left(\frac{1}{T}M_X H\left(\frac{\Theta'\Theta}{n}\right)H'M_X\right) \\
&\geq \mu_m\left(\frac{\Theta'\Theta}{n}\right)\mu_1\left(\frac{1}{T}M_X HH'M_X\right).
\end{aligned}$$

Now, we use that matrices AA' and $A'A$ share the same non-zero eigenvalues, and M_X is idempotent. Thus,

$$\mu_1\left(\frac{1}{T}M_X HH'M_X\right) = \mu_1\left(\frac{H'M_XH}{T}\right).$$

A.2.2 Proof of Proposition 1

a) The OLS estimator of β_i in matrix notation is $\hat{\beta}_i = \left(\tilde{X}_i'\tilde{X}_i\right)^{-1}\tilde{X}_i'\tilde{R}_i$, with $\tilde{R}_i = \boldsymbol{I}_i \odot R_i$. We get the vector of residuals $\hat{\varepsilon}_i = R_i - X_i\left(\tilde{X}_i'\tilde{X}_i\right)^{-1}\tilde{X}_i'\tilde{R}_i$. Then, we have $\bar{\varepsilon}_i = \boldsymbol{I}_i \odot \hat{\varepsilon}_i = M_{\tilde{X}_i}\tilde{R}_i = M_{\tilde{X}_i}\tilde{\varepsilon}_i$, where $\tilde{\varepsilon}_i = \boldsymbol{I}_i \odot \varepsilon_i$ and $M_{\tilde{X}_i} = I_T - P_{\tilde{X}_i}$, with $P_{\tilde{X}_i} = \tilde{X}_i\left(\tilde{X}_i'\tilde{X}_i\right)^{-1}\tilde{X}_i'$. Let us decompose $\sum_i \mathbf{1}_i^\chi \bar{\varepsilon}_i\bar{\varepsilon}_i'$ as:

$$\begin{aligned}
\sum_i \mathbf{1}_i^\chi \bar{\varepsilon}_i\bar{\varepsilon}_i' &= \sum_i \tilde{\varepsilon}_i\tilde{\varepsilon}_i' + \sum_i\left(\mathbf{1}_i^\chi - 1\right)\tilde{\varepsilon}_i\tilde{\varepsilon}_i' \\
&\quad - \sum_i \mathbf{1}_i^\chi\left(P_{\tilde{X}_i}\tilde{\varepsilon}_i\tilde{\varepsilon}_i' + \tilde{\varepsilon}_i\tilde{\varepsilon}_i'P_{\tilde{X}_i} - P_{\tilde{X}_i}\tilde{\varepsilon}_i\tilde{\varepsilon}_i'P_{\tilde{X}_i}\right).
\end{aligned}$$

The second term in the r.h.s. is a negative semi-definite matrix. Hence:

$$\mu_1\left(\frac{1}{nT}\sum_i \mathbf{1}_i^{\chi}\bar{\varepsilon}_i\bar{\varepsilon}_i'\right) \leq \mu_1\left(\frac{1}{nT}\sum_i \tilde{\varepsilon}_i\tilde{\varepsilon}_i' - \frac{1}{nT}\sum_i \mathbf{1}_i^{\chi}\left(P_{\tilde{X}_i}\tilde{\varepsilon}_i\tilde{\varepsilon}_i' + \tilde{\varepsilon}_i\tilde{\varepsilon}_i'P_{\tilde{X}_i} - P_{\tilde{X}_i}\tilde{\varepsilon}_i\tilde{\varepsilon}_i'P_{\tilde{X}_i}\right)\right).$$

Let us now use that the largest eigenvalue $\mu_1(A)$ of a symmetric matrix A corresponds to its operator norm $\|A\|_{op} = \sup_{x:\|x\|=1} \|Ax\|$, and that the operator norm (as any matrix norm) is such that $\|A+B\|_{op} \leq \|A\|_{op} + \|B\|_{op}$ for two matrices A and B. We have

$$\begin{aligned}
\mu_1\left(\frac{1}{nT}\sum_i \mathbf{1}_i^{\chi}\bar{\varepsilon}_i\bar{\varepsilon}_i'\right) &\leq \left\|\frac{1}{nT}\sum_i \tilde{\varepsilon}_i\tilde{\varepsilon}_i'\right\|_{op} + \left\|\frac{1}{nT}\sum_i \mathbf{1}_i^{\chi}P_{\tilde{X}_i}\tilde{\varepsilon}_i\tilde{\varepsilon}_i'\right\|_{op} \\
&\quad + \left\|\frac{1}{nT}\sum_i \mathbf{1}_i^{\chi}\tilde{\varepsilon}_i\tilde{\varepsilon}_i'P_{\tilde{X}_i}\right\|_{op} + \left\|\frac{1}{nT}\sum_i \mathbf{1}_i^{\chi}P_{\tilde{X}_i}\tilde{\varepsilon}_i\tilde{\varepsilon}_i'P_{\tilde{X}_i}\right\|_{op} \\
&\leq \mu_1\left(\frac{1}{nT}\sum_i \tilde{\varepsilon}_i\tilde{\varepsilon}_i'\right) + \|I_1\| + \|I_2\| + \|I_3\|, \qquad (11)
\end{aligned}$$

where $\|\cdot\|$ is the Frobenius norm and it is such that $\|A\|_{op} \leq \|A\|$, for any $T \times T$ matrix A (see, for example, Meyer (2000)). To control terms $\|I_j\|$, $j = 1, 2, 3$ in the r.h.s. of (11) we use the next Lemma, which is proved in Section A.2.3.

Lemma 1 *Under Assumptions A.1, A.3 and A.5, (i)* $\|I_1\| = O_p\left(\frac{\chi_{1,T}^2\chi_{2,T}}{T}\right)$, *(ii)* $\|I_2\| = O_p\left(\frac{\chi_{1,T}^2\chi_{2,T}}{T}\right)$ *and (iii)* $\|I_3\| = o_p\left(\frac{1}{T}\right)$, *when* $n, T \to \infty$.

From Equation (11) and Lemma 1, we get $\mu_1\left(\frac{1}{nT}\sum_i \mathbf{1}_i^{\chi}\bar{\varepsilon}_i\bar{\varepsilon}_i'\right) \leq \mu_1\left(\frac{1}{nT}\sum_i \tilde{\varepsilon}_i\tilde{\varepsilon}_i'\right) + O_p\left(\frac{\chi_{1,T}^2\chi_{2,T}}{T}\right)$. Under model $\mathcal{M}_1$, we have $\mu_1\left(\frac{1}{nT}\sum_i \tilde{\varepsilon}_i\tilde{\varepsilon}_i'\right) = O_p\left(C_{nT}^{-2}\right)$, if the $\tilde{\varepsilon}_{i,t} = I_{i,t}\varepsilon_{i,t}$ satisfy the weak time-series and cross-sectional correlation assumptions in Bai and Ng (2002b) and Ahn and Horenstein (2013). Then,

$$\xi = O_p\left(C_{nT}^{-2}\right) + O_p\left(\frac{\chi_{1,T}^2\chi_{2,T}}{T}\right) - g(n,T).$$

Since $g(n,T)\,C_{nT}^2 \to \infty$ and using Assumption A.5, we get $\xi = -g(n,T)(1 + o(1))$. Proposition 1a) follows.

b) Under model $\mathcal{M}_2$, by definition of $\bar{\varepsilon}_i$, we have:

$$\sum_i \mathbf{1}_i^{\chi} \bar{\varepsilon}_i \bar{\varepsilon}_i' = \sum_i M_{\tilde{X}_i} \tilde{\varepsilon}_i \tilde{\varepsilon}_i' M_{\tilde{X}_i} + \sum_i \left(\mathbf{1}_i^{\chi} - 1 \right) M_{\tilde{X}_i} \tilde{\varepsilon}_i \tilde{\varepsilon}_i' M_{\tilde{X}_i},$$

where $\tilde{\varepsilon}_i = \tilde{H}_i \theta_i + \tilde{u}_i$, with $\tilde{u}_i = \boldsymbol{I}_i \odot u_i$. By using $\|A+B\|_{op} \geq \|A\|_{op} - \|B\|_{op}$ and $\|A\|_{op} \leq \|A\|$, we have

$$\begin{aligned}
\mu_1 \left(\frac{1}{nT} \sum_i \mathbf{1}_i^{\chi} \bar{\varepsilon}_i \bar{\varepsilon}_i' \right) \geq & \left\| \frac{1}{nT} \sum_i M_{\tilde{X}_i} \tilde{H}_i \theta_i \theta_i' \tilde{H}_i' M_{\tilde{X}_i} \right\|_{op} - \left(\left\| \frac{1}{nT} \sum_i \left(1 - \mathbf{1}_i^{\chi} \right) M_{\tilde{X}_i} \tilde{\varepsilon}_i \tilde{\varepsilon}_i' M_{\tilde{X}_i} \right\|_{op} \right. \\
& + \left\| \frac{1}{nT} \sum_i M_{\tilde{X}_i} \tilde{H}_i \theta_i \tilde{u}_i' M_{\tilde{X}_i} \right\|_{op} + \left\| \frac{1}{nT} \sum_i M_{\tilde{X}_i} \tilde{u}_i \theta_i' \tilde{H}_i' M_{\tilde{X}_i} \right\|_{op} \\
& \left. + \left\| \frac{1}{nT} \sum_i M_{\tilde{X}_i} \tilde{u}_i \tilde{u}_i' M_{\tilde{X}_i} \right\|_{op} \right) \\
\geq & \; \mu_1 \left(\frac{1}{nT} \sum_i M_{\tilde{X}_i} \tilde{H}_i \theta_i \theta_i' \tilde{H}_i' M_{\tilde{X}_i} \right) - \left(\|I_4\| + \|I_5\| + \|I_6\| + \|I_7\|_{op} \right). \quad (12)
\end{aligned}$$

Proposition 1(b) follows from the next Lemma, which is proved in Section A.2.4.

Lemma 2 *Under Assumptions A.2-A.5, (i)* $\|I_4\| = O_p(T^{-\bar{b}})$, *for any* $\bar{b} > 0.$, *(ii)* $\|I_5\| = O_p\left(\frac{1}{\sqrt{n}}\right)$, *(iii)* $\|I_6\| = O_p\left(\frac{1}{\sqrt{n}}\right)$ *and (iv)* $\|I_7\|_{op} = O_p\left(C_{n,T}^{-2} + \frac{\chi_{1,T}^4 \chi_{2,T}^2}{T}\right)$, *when* $n, T \to \infty$.

From Equation (12) and Lemma 2, we get $\mu_1 \left(\frac{1}{nT} \sum_i \mathbf{1}_i^{\chi} \bar{\varepsilon}_i \bar{\varepsilon}_i' \right) \geq \mu_1 \left(\frac{1}{nT} \sum_i M_{\tilde{X}_i} \tilde{H}_i \theta_i \theta_i' \tilde{H}_i' M_{\tilde{X}_i} \right)$ $+ o_p(1)$. Under Assumption 1, the result follows.

A.2.3 Proof of Lemma 1

Term depending on I_1. Let us define the information sets $\mathcal{J}_i = \left\{x_{i,\underline{T}}, I_{i,\underline{T}}, \gamma_i\right\}$ for asset i and $\mathcal{J} = \bigcup_{i=1}^{n} \mathcal{J}_i$. By using the properties of the trace operator, we have:

$$
\begin{aligned}
E\left[\|I_1\|^2 |\mathcal{J}\right] &= E\left[\frac{1}{n^2T^2} Tr\left[\sum_{i,j} \mathbf{1}_i^\chi \mathbf{1}_j^\chi P_{\tilde{X}_i} \tilde{\varepsilon}_i \tilde{\varepsilon}_i' \tilde{\varepsilon}_j \tilde{\varepsilon}_j' P_{\tilde{X}_j}\right] |\mathcal{J}\right] \\
&= E\left[\frac{1}{n^2T^2} \sum_{i,j} \mathbf{1}_i^\chi \mathbf{1}_j^\chi \tilde{\varepsilon}_j' P_{\tilde{X}_j} P_{\tilde{X}_i} \tilde{\varepsilon}_i \tilde{\varepsilon}_i' \tilde{\varepsilon}_j |\mathcal{J}\right] \\
&= \frac{1}{n^2T^2} \sum_{i,j} \mathbf{1}_i^\chi \mathbf{1}_j^\chi E\left[\tilde{\varepsilon}_j' P_{\tilde{X}_j} P_{\tilde{X}_i} \tilde{\varepsilon}_i \left(\tilde{\varepsilon}_i' \tilde{\varepsilon}_j\right) |\mathcal{J}_i, \mathcal{J}_j\right].
\end{aligned}
$$

To ease notation $x_{i,t}$ is now treated as a scalar. By using that $\tilde{\varepsilon}_i' \tilde{\varepsilon}_j = \sum_t I_{ij,t} \varepsilon_{i,t} \varepsilon_{j,t}$ where $I_{ij,t} = I_{i,t} I_{j,t}$ for $i, j = 1, ..., n$, and $\tilde{\varepsilon}_j' P_{\tilde{X}_j} P_{\tilde{X}_i} \tilde{\varepsilon}_i = \hat{Q}_{x,i}^{-1} \hat{Q}_{x,j}^{-1} \left(\sum_t I_{ij,t} x_{i,t} x_{j,t}\right) \left(\frac{1}{T_i} \sum_t I_{i,t} x_{i,t} \varepsilon_{i,t}\right) \left(\frac{1}{T_j} \sum_t I_{j,t} x_{j,t} \varepsilon_{j,t}\right)$, we get

$$
\begin{aligned}
E\left[\|I_1\|^2 |\mathcal{J}\right] &\leq \frac{1}{n^2T^3} \sum_{i,j} \sum_{t_1,t_2,t_3} \mathbf{1}_i^\chi \mathbf{1}_j^\chi \tau_{i,T} \tau_{j,T} I_{i,t_1} I_{j,t_2} I_{ij,t_3} \hat{Q}_{x,i}^{-1} \hat{Q}_{x,j}^{-1} \\
&\quad \left(\frac{1}{T} \sum_t I_{ij,t} x_{i,t} x_{j,t}\right) x_{i,t_1} x_{j,t_2} E\left[\varepsilon_{i,t_1} \varepsilon_{j,t_2} \varepsilon_{i,t_3} \varepsilon_{j,t_3} |\mathcal{J}_i, \mathcal{J}_j\right].
\end{aligned}
$$

From the definition of condition number CN, $\|\hat{Q}_{x,i}^{-1}\|^2 = Tr\left(\hat{Q}_{x,i}^{-2}\right) = \sum_{k=1}^{d} \mu_{k,i}^{-2} \leq dCN\left(\hat{Q}_{x,i}\right)^4$, where the $\mu_{k,i}$ are the eigenvalues of matrix $\hat{Q}_{x,i}$, we use that $\mu_{1,i}\left(\hat{Q}_{x,i}\right) \geq 1$, which implies $\mathbf{1}_i^\chi \|\hat{Q}_{x,i}^{-1}\| \leq C\chi_{1,T}^2$. Moreover, by $\mathbf{1}_i^\chi \tau_{i,T} \leq \chi_{2,T}$ and Assumption A.3, we deduce that

$$
E\left[\|I_1\|^2 |\mathcal{J}\right] \leq \frac{1}{n^2T^3} C\chi_{1,T}^4 \chi_{2,T}^2 \sum_{i,j} \sum_{t_1,t_2,t_3} \left|E\left[\varepsilon_{i,t_1} \varepsilon_{j,t_2} \varepsilon_{i,t_3} \varepsilon_{j,t_3} \left|x_{i,\underline{T}}, x_{j,\underline{T}}, \gamma_i, \gamma_j\right.\right]\right|.
$$

From Assumption A.1a), we have $E\left[\|I_1\|^2 |\mathcal{J}\right] \leq \frac{1}{T^2} C\chi_{1,T}^4 \chi_{2,T}^2$, which implies $\|I_1\| = O_p\left(\frac{\chi_{1,T}^2 \chi_{2,T}}{T}\right)$.

Term depending on I_2. The result follows by similar arguments used to prove that $\|I_1\| = O_p\left(\frac{\chi_{1,T}^2 \chi_{2,T}}{T}\right)$.

Term depending on I_3. By the properties of trace operator, we have

$$\begin{aligned} E\left[\|I_3\|^2 |\mathcal{J}\right] &= E\left[\frac{1}{n^2T^2}Tr\left[\sum_{i,j}\mathbf{1}_i^\chi\mathbf{1}_j^\chi P_{\tilde{X}_i}\tilde{\varepsilon}_i\tilde{\varepsilon}_i' P_{\tilde{X}_i}P_{\tilde{X}_j}\tilde{\varepsilon}_j\tilde{\varepsilon}_j' P_{\tilde{X}_j}\right]|\mathcal{J}\right] \\ &= E\left[\frac{1}{n^2T^2}\sum_{i,j}\mathbf{1}_i^\chi\mathbf{1}_j^\chi\left(\tilde{\varepsilon}_j' P_{\tilde{X}_j}P_{\tilde{X}_i}\tilde{\varepsilon}_i\right)^2|\mathcal{J}\right] \\ &= \frac{1}{n^2T^2}\sum_{i,j}\mathbf{1}_i^\chi\mathbf{1}_j^\chi E\left[\left(\tilde{\varepsilon}_j' P_{\tilde{X}_j}P_{\tilde{X}_i}\tilde{\varepsilon}_i\right)^2|\mathcal{J}_i,\mathcal{J}_j\right]. \end{aligned}$$

By using that $\tilde{\varepsilon}_j' P_{\tilde{X}_j}P_{\tilde{X}_i}\tilde{\varepsilon}_i = \hat{Q}_{x,i}^{-1}\hat{Q}_{x,j}^{-1}\left(\sum_t I_{ij,t}x_{i,t}x_{j,t}\right)\left(\frac{1}{T_i}\sum_t I_{i,t}x_{i,t}\varepsilon_{i,t}\right)\left(\frac{1}{T_j}\sum_t I_{j,t}x_{j,t}\varepsilon_{j,t}\right)$, we get

$$\begin{aligned} E\left[\|I_3\|^2 |\mathcal{J}\right] \leq\; & \frac{1}{n^2T^4}\sum_{i,j}\sum_{t_1,t_2,t_3,t_4}\mathbf{1}_i^\chi\mathbf{1}_j^\chi\tau_{i,T}^2\tau_{j,T}^2 I_{i,t_1}I_{i,t_2}I_{j,t_3}I_{j,t_4}\hat{Q}_{x,i}^{-2}\hat{Q}_{x,j}^{-2} \\ & \left(\frac{1}{T}\sum_t I_{ij,t}x_{i,t}x_{j,t}\right)^2 x_{i,t_1}x_{i,t_2}x_{j,t_3}x_{j,t_4}E\left[\varepsilon_{i,t_1}\varepsilon_{i,t_2}\varepsilon_{j,t_3}\varepsilon_{j,t_4}|\mathcal{J}_i,\mathcal{J}_j\right]. \end{aligned}$$

From $\mathbf{1}_i^\chi\|\hat{Q}_{x,i}^{-1}\| \leq C\chi_{1,T}^2$, $\mathbf{1}_i^\chi\tau_{i,T} \leq \chi_{2,T}$ and Assumption A.3, we deduce that

$$E\left[\|I_3\|^2 |\mathcal{J}_i,\mathcal{J}_j\right] = \frac{1}{n^2T^4}C\chi_{1,T}^8\chi_{2,T}^4\sum_{i,j}\sum_{t_1,t_2,t_3,t_4}\left|E\left[\varepsilon_{i,t_1}\varepsilon_{i,t_2}\varepsilon_{j,t_3}\varepsilon_{j,t_4}\left|x_{i,\underline{T}},x_{j,\underline{T}},\gamma_i,\gamma_j\right.\right]\right|.$$

Then, by Assumption A.1b) the result follows.

A.2.3 Proof of Lemma 2

Term depending on I_4. We have:

$$\|I_4\|_{op} \leq \frac{1}{nT}\sum_i(1-\mathbf{1}_i^\chi)\left\|M_{\tilde{X}_i}\tilde{\varepsilon}_i\tilde{\varepsilon}_i' M_{\tilde{X}_i}\right\|_{op} \leq \frac{1}{nT}\sum_i(1-\mathbf{1}_i^\chi)\left\|\tilde{\varepsilon}_i\tilde{\varepsilon}_i'\right\|_{op},$$

where the second inequality is because matrix $M_{\tilde{X}_i}$ is idempotent. Moreover, by using $\left\|\tilde{\varepsilon}_i\tilde{\varepsilon}_i'\right\|_{op} \leq \left\|\tilde{\varepsilon}_i\tilde{\varepsilon}_i'\right\| = \tilde{\varepsilon}_i'\tilde{\varepsilon}_i$, we get:

$$\|I_4\|_{op} \leq \frac{1}{n}\sum_i(1-\mathbf{1}_i^\chi)\left(\frac{1}{T}\sum_t I_{i,t}\varepsilon_{i,t}^2\right).$$

By the Cauchy-Schwarz inequality we have:

$$E\left[\|I_4\|_{op}\right] \leq \frac{1}{n}\sum_i P\left[\mathbf{1}_i^{\chi}=0\right]^{1/2}\left(E\left[\left(\frac{1}{T}\sum_t I_{i,t}\varepsilon_{i,t}^2\right)^2\right]\right)^{1/2}.$$

Lemma 7 of GOS states that $P\left[\mathbf{1}_i^{\chi}=0\right]=O(T^{-\bar{b}})$, for any $\bar{b}>0$. This unconditional probability is independent of i since the indices (γ_i) are i.i.d.. From Assumption A.2 f), we have $E\left[\left(\frac{1}{T}\sum_t I_{i,t}\varepsilon_{i,t}^2\right)^2\right] \leq M$ for all i and a constant M. Thus, $E\left[\|I_4\|_{op}\right]=O(T^{-\bar{b}})$, for any $\bar{b}>0$.

Term depending on I_5. By the definition of $M_{\tilde{X}_i}$, we have $M_{\tilde{X}_i}\tilde{H}_i\theta_i\tilde{u}_i' M_{\tilde{X}_i} = \tilde{H}\theta_i\tilde{u}_i' - \tilde{H}_i\theta_i\tilde{u}_i' P_{\tilde{X}_i} - P_{\tilde{X}_i}\tilde{H}_i\theta_i\tilde{u}_i' + P_{\tilde{X}_i}\tilde{H}_i\theta_i\tilde{u}_i' P_{\tilde{X}_i}$. From the properties of the norm operator, we deduce

$$\begin{aligned}\|I_5\| &\leq \left\|\frac{1}{nT}\sum_i \tilde{H}_i\theta_i\tilde{u}_i'\right\| + \left\|\frac{1}{nT}\sum_i \tilde{H}_i\theta_i\tilde{u}_i' P_{\tilde{X}_i}\right\| + \left\|\frac{1}{nT}\sum_i P_{\tilde{X}_i}\tilde{H}_i\theta_i\tilde{u}_i'\right\| + \left\|\frac{1}{nT}\sum_i P_{\tilde{X}_i}\tilde{H}_i\theta_i\tilde{u}_i' P_{\tilde{X}_i}\right\| \\ &= \|I_{51}\| + \|I_{52}\| + \|I_{53}\| + \|I_{54}\|.\end{aligned}$$

By the properties of the trace operator, we have:

$$\begin{aligned}E\left[\|I_{51}\|^2 \,\middle|\, h_{\underline{T}}, \mathcal{J}\right] &= E\left[\frac{1}{n^2T^2}Tr\left[\sum_{i,j}\tilde{H}_i\theta_i\tilde{u}_i'\tilde{u}_j\theta_j'\tilde{H}_j'\right] \,\middle|\, h_{\underline{T}}, \mathcal{J}\right] \\ &= \frac{1}{n^2T^2}\sum_{i,j}E\left[\theta_j'\tilde{H}_i'\tilde{H}_i\theta_i\tilde{u}_i'\tilde{u}_j \,\middle|\, h_{\underline{T}}, \mathcal{J}_i, \mathcal{J}_j\right],\end{aligned}$$

where $\theta_j'\tilde{H}_j{}'\tilde{H}_i\theta_i = \sum_t I_{ij,t}h_t'\theta_i\theta_j' h_t$ and $\tilde{u}_i'\tilde{u}_j = \sum_t I_{ij,t}u_{i,t}u_{j,t}$. To ease notation θ_i is now treated as a scalar. Then,

$$E\left[\|I_{51}\|^2 \,\middle|\, h_{\underline{T}}, \mathcal{J}\right] \leq \frac{1}{n^2T}\sum_{i,j}\sum_t I_{ij,t}\theta_i\theta_j\left(\frac{1}{T}\sum_t h_t'h_t\right)E\left[u_{i,t}u_{j,t} \,\middle|\, h_{\underline{T}}, \mathcal{J}_i, \mathcal{J}_j\right].$$

By Assumptions A.2a) and A.4, we have $E\left[\|I_{51}\|^2 \,\middle|\, h_{\underline{T}}, \mathcal{J}\right] \leq \frac{1}{n}C$, which implies $\|I_{51}\| = O_p\left(\frac{1}{\sqrt{n}}\right)$. Similarly, by applying the properties of the trace operator and expectation, we have:

$$\begin{aligned}E\left[\|I_{52}\|^2 \,\middle|\, h_{\underline{T}}, \mathcal{J}\right] &= \frac{1}{n^2T^2}\sum_{i,j}E\left[\theta_j'\tilde{H}_j'\tilde{H}_i\theta_i\tilde{u}_i' P_{\tilde{X}_i}P_{\tilde{X}_j}\tilde{u}_j \,\middle|\, h_{\underline{T}}, \mathcal{J}_i, \mathcal{J}_j\right] \\ &\leq \frac{1}{n^2T^2}\sum_{i,j}\sum_{t_1,t_2}\tau_{i,T}\tau_{j,T}I_{i,t_1}I_{j,t_2}\theta_i\theta_j\hat{Q}_{x,i}^{-1}\hat{Q}_{x,j}^{-1}\left(\frac{1}{T}\sum_t h_t'h_t\right)\left(\frac{1}{T}\sum_t I_{ij,t}x_{i,t}x_{j,t}\right) \\ &\quad x_{i,t_1}x_{j,t_2}E\left[u_{i,t_1}u_{j,t_2} \,\middle|\, h_{\underline{T}}, \mathcal{J}_i, \mathcal{J}_j\right].\end{aligned}$$

From $\tau_{i,T} \leq \chi_{2,T}$ and $\left\|\hat{Q}_{x,i}^{-1}\right\| \leq C\chi_{1,T}^2$, and Assumptions A.3, A.4 and A.2b), $E\left[\|I_{52}\|^2 \left| h_{\underline{T}}, \mathcal{J}\right.\right] \leq \frac{1}{nT}C\chi_{1,T}^4\chi_{2,T}^2$, which implies $\|I_{52}\| = O_p\left(\frac{\chi_{1,T}^2\chi_{2,T}}{\sqrt{nT}}\right)$. Moreover, we get

$$
\begin{aligned}
E\left[\|I_{53}\|^2 \left| h_{\underline{T}}, \mathcal{J}\right.\right] &= \frac{1}{n^2T^2}\sum_{i,j} E\left[\theta_j'\tilde{H}_j'P_{\tilde{X}_j}P_{\tilde{X}_i}\tilde{H}_i\theta_i\tilde{u}_i'\tilde{u}_j \left| h_{\underline{T}}, \mathcal{J}_i, \mathcal{J}_j\right.\right] \\
&\leq \frac{1}{n^2T}\sum_{i,j}\sum_t \tau_{i,T}\tau_{j,t}I_{ij,t}\theta_i\theta_j\hat{Q}_{x,i}^{-1}\hat{Q}_{x,j}^{-1}\left(\frac{1}{T}\sum_t I_{ij,t}x_{i,t}x_{j,t}\right) \\
&\quad \left(\frac{1}{T}\sum_t I_{i,t}x_{i,t}h_t\right)\left(\frac{1}{T}\sum_t I_{j,t}x_{j,t}h_t\right)E\left[u_{i,t}u_{j,t}\left| h_{\underline{T}}, \mathcal{J}_i, \mathcal{J}_j\right.\right].
\end{aligned}
$$

By using that $\tau_{i,T} \leq \chi_{2,T}$, $\left\|\hat{Q}_{x,i}^{-1}\right\| \leq C\chi_{1,T}^2$, and Assumptions A.2a), A.3, and A.4, $E\left[\|I_{53}\|^2 \left| h_{\underline{T}}, \mathcal{J}\right.\right] \leq \frac{1}{n}C\chi_{1,T}^4\chi_{2,T}^2$, and $\|I_{53}\| = O_p\left(\frac{\chi_{1,T}^2\chi_{2,T}}{\sqrt{n}}\right)$. Finally, we have:

$$
\begin{aligned}
E\left[\|I_{54}\|^2 \left| h_{\underline{T}}, \mathcal{J}\right.\right] &= \frac{1}{n^2T^2}\sum_{i,j} E\left[\theta_j'\tilde{H}_j'P_{\tilde{X}_j}P_{\tilde{X}_i}\tilde{H}_i\theta_i\tilde{u}_i'P_{\tilde{X}_i}P_{\tilde{X}_j}\tilde{u}_j \left| h_{\underline{T}}, \mathcal{J}_i, \mathcal{J}_j\right.\right] \\
&\leq \frac{1}{n^2T^2}\sum_{i,j}\sum_{t_1,t_2} \tau_{i,T}^2\tau_{j,t}^2 I_{i,t_1}I_{j,t_2}\theta_i\theta_j\hat{Q}_{x,i}^{-2}\hat{Q}_{x,j}^{-2}\left(\frac{1}{T}\sum_t I_{ij,t}x_{i,t}x_{j,t}\right)^2 \\
&\quad \left(\frac{1}{T}\sum_t I_{i,t}x_{i,t}h_t\right)\left(\frac{1}{T}\sum_t I_{j,t}x_{j,t}h_t\right)x_{i,t_1}x_{j,t_2}E\left[u_{i,t_1}u_{j,t_2}\left| h_{\underline{T}}, \mathcal{J}_i, \mathcal{J}_j\right.\right],
\end{aligned}
$$

then, by $\tau_{i,T} \leq \chi_{2,T}$, $\left\|\hat{Q}_{x,i}^{-1}\right\| \leq C\chi_{1,T}^2$, and Assumptions A.2b), A.3, and A.4, $E\left[\|I_{54}\|^2 \left| h_{\underline{T}}, \mathcal{J}\right.\right] \leq \frac{1}{nT}C\chi_{1,T}^8\chi_{2,T}^4$, and $\|I_{54}\| = O_p\left(\frac{\chi_{1,T}^4\chi_{2,T}^2}{\sqrt{nT}}\right)$. The results obtained for terms I_{51}, I_{52}, I_{53} and I_{54} imply that $\|I_5\| = O_p\left(\frac{1}{\sqrt{n}}\right)$.

Term depending on I_6. The result follows by similar arguments used to prove that $\|I_5\| = O_p\left(\frac{1}{\sqrt{n}}\right)$.

Term depending on I_7. By the definition of $M_{\tilde{X}_i}$, we have $M_{\tilde{X}_i}\tilde{u}_i\tilde{u}_i'M_{\tilde{X}_i} = \tilde{u}_i\tilde{u}_i' - \tilde{u}_i\tilde{u}_i'P_{\tilde{X}_i} - P_{\tilde{X}_i}\tilde{u}_i\tilde{u}_i' + P_{\tilde{X}_i}\tilde{u}_i\tilde{u}_i'P_{\tilde{X}_i}$. From the properties of the norm operator, we deduce

$$
\begin{aligned}
\|I_7\|_{op} &\leq \left\|\frac{1}{nT}\sum_i \tilde{u}_i\tilde{u}_i'\right\|_{op} + \left\|\frac{1}{nT}\sum_i \tilde{u}_i\tilde{u}_i'P_{\tilde{X}_i}\right\|_{op} + \left\|\frac{1}{nT}\sum_i P_{\tilde{X}_i}\tilde{u}_i\tilde{u}_i'\right\|_{op} + \left\|\frac{1}{nT}\sum_i P_{\tilde{X}_i}\tilde{u}_i\tilde{u}_i'P_{\tilde{X}_i}\right\|_{op} \\
&\leq \|I_{71}\|_{op} + \|I_{72}\| + \|I_{73}\| + \|I_{74}\|.
\end{aligned}
$$

If the $\tilde{u}_{i,t} = I_{i,t}u_{i,t}$ satisfy the weak time-series and cross-sectional correlation assumptions in Bai and Ng (2002b) and Ahn and Horenstein (2013), we have $\|I_{71}\|_{op} \leq \mu_1 (I_{71}) = O_p\left(C_{n,T}^{-2}\right)$. Moreover,

$$\begin{aligned} E\left[\|I_{72}\|^2 \left|h_{\underline{T}}, \mathcal{J}\right.\right] &= \frac{1}{n^2T^2}\sum_{i,j} E\left[\tilde{u}_j'\tilde{u}_i\tilde{u}_i'P_{\tilde{X}_i}P_{\tilde{X}_j}\tilde{u}_j \left|h_{\underline{T}}, \mathcal{J}_i, \mathcal{J}_j\right.\right] \\ &\leq \frac{1}{n^2T^3}\sum_{i,j}\sum_{t_1,t_2,t_3} \tau_{i,T}\tau_{j,T}I_{i,t}I_{j,t}I_{ij,t}\hat{Q}_{x,i}^{-1}\hat{Q}_{x,j}^{-1} \\ &\quad \left(\frac{1}{T}\sum_t I_{ij,t}x_{i,t}x_{j,t}\right)x_{i,t_1}x_{j,t_2}E\left[u_{i,t_1}u_{j,t_2}u_{i,t_3}u_{j,t_3}\left|h_{\underline{T}}, \mathcal{J}_i, \mathcal{J}_j\right.\right]. \end{aligned}$$

The bound $E\left[\|I_{72}\|^2 \left|h_{\underline{T}}, \mathcal{J}\right.\right] \leq \frac{1}{T^2}C\chi_{1,T}^4\chi_{2,T}^2$ follows by $\tau_{i,T} \leq \chi_{2,T}$, $\left\|\hat{Q}_{x,i}^{-1}\right\| \leq C\chi_{1,T}^2$, and Assumptions A.2d) and A.3. Thus, $\|I_{72}\| = O_p\left(\frac{\chi_{1,T}^2\chi_{2,T}}{T}\right)$. By applying similar arguments, we get $E\left[\|I_{73}\|^2 \left|h_{\underline{T}}, \mathcal{J}\right.\right] \leq \frac{1}{T^2}C\chi_{1,T}^4\chi_{2,T}^2$ and $\|I_{73}\| = O_p\left(\frac{\chi_{1,T}^2\chi_{2,T}}{T}\right)$. Furthermore, by using similar arguments for I_3, from $\tau_{i,T} \leq \chi_{2,T}$, $\left\|\hat{Q}_{x,i}^{-1}\right\| \leq C\chi_{1,T}^2$, and Assumptions A.2d), A.3, we get $E\left[\|I_{74}\|^2 \left|h_{\underline{T}}, \mathcal{J}\right.\right] \leq \frac{1}{T^2}C\chi_{1,T}^8\chi_{2,T}^4$ and $\|I_{74}\| = O_p\left(\frac{\chi_{1,T}^4\chi_{2,T}^2}{T}\right)$. Finally, we deduce that $\|I_7\|_{op} = O_p\left(C_{n,T}^{-2} + \frac{\chi_{1,T}^4\chi_{2,T}^2}{T}\right)$.

Appendix 3: Link with Stock and Watson (2002)

We consider the EM algorithm proposed by Stock and Watson (2002b):

$$\tilde{\tilde{\varepsilon}}_{i,t} = \begin{cases} \hat{\varepsilon}_{i,t}, & \text{if } I_{i,t} = 1, \\ \hat{\theta}_i\hat{h}_t, & \text{if } I_{i,t} = 0. \end{cases}$$

The statistic is $\xi = \mu_1\left(\frac{\tilde{\tilde{\varepsilon}}\tilde{\tilde{\varepsilon}}'}{nT}\right) - \frac{1}{nT}\sum_i\sum_t (1 - I_{i,t})\left(\hat{\theta}_i\hat{h}_t\right)^2 - g(n, T)$. Below we show that ξ is the difference of the EM criteria under the two models. Comparing the two test statistics gives the following link: $\frac{1}{nT}\sum_i\sum_t (1 - I_{i,t})\left(\hat{\theta}_i\hat{h}_t\right)^2 = \frac{1}{nT}\left\|\tilde{\varepsilon} - \tilde{\tilde{\varepsilon}}\right\|^2$.

To study the EM algorithm, we work as if the true error terms $\varepsilon_{i,t}$ are observed when $I_{i,t} = 1$. This error is replaced by the residual $\hat{\varepsilon}_{i,t}$. We consider the jth iteration of the algorithm. Let $\tilde{\zeta} = \left(\tilde{\Theta}, \tilde{H}\right)$ denotes the estimates of Θ and H obtained from the $(j-1)$th iteration, and let $Q\left(\zeta, \tilde{\zeta}\right) = E_{\tilde{\zeta}}\left[\mathcal{L}(\zeta)\left|\varepsilon\right.\right]$, where $\mathcal{L}(\zeta)=\frac{1}{nT}\sum_i\sum_t\left(\varepsilon^*_{i,t}-\theta_i h_t\right)^2$, and $E_{\tilde{\zeta}}[\cdot|\varepsilon]$ denotes conditional expectation given the panel of observations under parameter $\tilde{\zeta}$. We study $Q\left(\zeta,\tilde{\zeta}\right)$ under the two models. Under both $\mathcal{M}_1$ and $\mathcal{M}_2$, we consider a pseudo model for the innovations such that $u_{i,t}\sim i.i.d.\left(0,\sigma^2_{i,t}\right)$

- Under $\mathcal{M}_1$: we get

$$Q_0\left(\zeta,\tilde{\zeta}\right)=E\left[\frac{1}{nT}\sum_i\sum_t\left(\varepsilon^*_{i,t}\right)^2|\varepsilon\right]=\frac{1}{nT}\sum_i\sum_t E\left[\left(\varepsilon^*_{i,t}\right)^2|\varepsilon\right].$$

We have

$$E\left[\varepsilon^*_{i,t}|\varepsilon\right]=\begin{cases}\varepsilon_{i,t}, & \text{if } I_{i,t}=1,\\ 0, & \text{if } I_{i,t}=0,\end{cases}\text{ and } V\left[\varepsilon^*_{i,t}|\varepsilon\right]=\begin{cases}0, & \text{if } I_{i,t}=1,\\ \sigma^2_{i,t}, & \text{if } I_{i,t}=0.\end{cases}$$

and $E\left[\left(\varepsilon^*_{i,t}\right)^2|\varepsilon\right]=I_{i,t}\varepsilon^2_{i,t}+\left(1-I_{i,t}\right)\sigma^2_{i,t}$. Thus,

$$Q_0=Q_0\left(\zeta,\tilde{\zeta}\right)=\frac{1}{nT}\sum_i\sum_t I_{i,t}\varepsilon^2_{i,t}+\frac{1}{nT}\sum_i\sum_t\left(1-I_{i,t}\right)\sigma^2_{i,t}.$$

- Under $\mathcal{M}_2$: we get

$$\begin{aligned}Q_1\left(\zeta,\tilde{\zeta}\right) &= E_{\tilde{\zeta}}\left[\frac{1}{nT}\sum_i\sum_t\left(\varepsilon^*_{i,t}-\theta_i h_t\right)^2|\varepsilon\right]\\ &= \frac{1}{nT}\sum_i\sum_t E_{\tilde{\zeta}}\left[\left(\varepsilon^*_{i,t}-\theta_i h_t\right)^2|\varepsilon\right]\\ &= \frac{1}{nT}\sum_i\sum_t E_{\tilde{\zeta}}\left[\left(\varepsilon^*_{i,t}-E_{\tilde{\zeta}}\left[\varepsilon^*_{i,t}|\varepsilon\right]+E_{\tilde{\zeta}}\left[\varepsilon^*_{i,t}|\varepsilon\right]-\theta_i h_t\right)^2|\varepsilon\right]\\ &= \frac{1}{nT}\sum_i\sum_t V_{\tilde{\zeta}}\left[\varepsilon^*_{i,t}|\varepsilon\right]+\frac{1}{nT}\sum_i\sum_t\left(E_{\tilde{\zeta}}\left[\varepsilon^*_{i,t}|\varepsilon\right]-\theta_i h_t\right)^2.\end{aligned}$$

We have

$$\tilde{\varepsilon}_{i,t}:=E_{\tilde{\zeta}}\left[\varepsilon^*_{i,t}|\varepsilon\right]=\begin{cases}\varepsilon_{i,t}, & \text{if } I_{i,t}=1,\\ \tilde{\theta}_i\tilde{h}_t, & \text{if } I_{i,t}=0,\end{cases}\text{ and } V\left[\varepsilon^*_{i,t}|\varepsilon\right]=\begin{cases}0, & \text{if } I_{i,t}=1,\\ \sigma^2_{i,t}, & \text{if } I_{i,t}=0.\end{cases}$$

Thus, $Q_1\left(\zeta,\tilde{\zeta}\right)=\frac{1}{nT}\sum_i\sum_t\left(\tilde{\tilde{\varepsilon}}_{i,t}-\theta_i h_t\right)^2+\frac{1}{nT}\sum_i\sum_t\left(1-I_{i,t}\right)\sigma^2_{i,t}$, and the values of ζ that minimize $Q_1\left(\zeta,\tilde{\zeta}\right)$ can be calculated by $\min_\zeta\frac{1}{nT}\sum_i\sum_t\left(\tilde{\tilde{\varepsilon}}_{i,t}-\theta_i h_t\right)^2$. This minimization problem

reduces to the usual PCA on data $\tilde{\tilde{\varepsilon}}$: $\min_{\zeta} \frac{1}{nT} \sum_i \sum_t \left(\tilde{\tilde{\varepsilon}}_{i,t} - \theta_i h_t\right)^2 = \frac{1}{nT} \sum_i \sum_t \tilde{\tilde{\varepsilon}}^2_{i,t} - \mu_1 \left(\frac{\tilde{\varepsilon}\tilde{\varepsilon}'}{nT}\right)$.
Therefore, at convergence with $\hat{\zeta} = \tilde{\zeta}$, we have

$$\begin{aligned} Q_1\left(\hat{\zeta}, \tilde{\zeta}\right) &= \frac{1}{nT} \sum_i \sum_t \tilde{\tilde{\varepsilon}}^2_{i,t} - \mu_1 \left(\frac{\tilde{\varepsilon}\tilde{\varepsilon}'}{nT}\right) + \frac{1}{nT} \sum_i \sum_t \left(1 - I_{i,t}\right) \sigma^2_{i,t} \\ &= \frac{1}{nT} \sum_i \sum_t I_{i,t} \varepsilon^2_{i,t} + \frac{1}{nT} \sum_i \sum_t \left(1 - I_{i,t}\right) \left(\hat{\theta}_i \hat{h}_t\right)^2 \\ &\quad -\mu_1 \left(\frac{\tilde{\tilde{\varepsilon}}\tilde{\tilde{\varepsilon}}'}{nT}\right) + \frac{1}{nT} \sum_i \sum_t \left(1 - I_{i,t}\right) \sigma^2_{i,t}. \end{aligned}$$

Finally, the difference of the two EM criterias is

$$Q_0 - Q_1\left(\hat{\zeta}, \hat{\zeta}\right) = \mu_1 \left(\frac{\tilde{\tilde{\varepsilon}}\tilde{\tilde{\varepsilon}}'}{nT}\right) - \frac{1}{nT} \sum_i \sum_t \left(1 - I_{i,t}\right) \left(\hat{\theta}_i \hat{h}_t\right)^2,$$

which gives the interpretation of the test statistic.