Identification of an Idea of Genius :

The invention of the present value in 1202

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According to Kac (1985), *there are two kinds of geniuses: the "ordinary" and the "magicians." An ordinary genius is a fellow whom you and I would be just as good as, if we were only many times better. There is no mystery as to how his mind works. Once we understand what they've done, we feel certain that we, too, could have done it. It is different with the magicians. Even after we understand what they have done it is completely dark.*

Following Ralph Waldo Emerson such genius appear because *when Nature has work to be done, she creates a genius to dot it* — Emerson (1841). This seems clear even if we never will know if the apparition of the genius is pertinent to Nature or to Nurture.

But in fact, this is not the point. How can one be sure that we are facing a magician or, more exactly, an idea of magician. After all, to qualify and idea of genius is essentially subjective. Let us take a simple example : Scipione del Ferro solution of the cubic equation[1]. First it was not very difficult to go from the standard third order equation — $x^3 + ax^2 + bx + c = 0$ — to the reduce form $y^3 = \alpha y + \beta$ by the change of variable $y = x - a/3$. What seems magic, is to try to change y for $u + v$, because it's difficult to see how to go from one variable to two. One can only expects to find a way. But strikingly, it worked since one obtains $(u + v)^3 = \alpha(u + v) + \beta$ which, after expansion, leads to a system of two non linear equations $u^3 + v^3 = \beta$ and $3uv = \alpha$. Eliminating v and substituting the resulting equation in the reduce form, one obtains $u^6 - \beta u^3 + (\alpha/3)^2 = 0$, which is a second order equation in disguise, if one set $z = u^3$.

The substitution of the sum of two unknowns for one could look as magic, but after reflection, as underlined by Blank (1999), it could be derived from the observation of the solution of the second order equation whose solution has two parts, and from the incorrect solutions of Paolo Gerardi and Dardi of Pisa which could be written as $u + \sqrt{w}$ and $u + \sqrt[3]{w}$.

There are a few theories which try to qualify what is an idea of genius. As for many subject we must refer to Kant (2000), whom conception can be translated from the one who produces the idea to the idea itself: what is brilliant is what is non-imitative. The most part of our ideas are reproductives : when confronted with a new problem, we try to start from a seemingly similar problem whose solution we know. Then we try to adapt this solution to the real problem posed. This is imitation and as clever

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[1]Del Ferro never published his solution perhaps because regularly the Bologna University position he holds was was regularly put in concourse. Dying, he legates his finding to his son-in-law Hannibal Nave and one of his student Antonio Fior. In turn in competition for an university position, this last one was opposed to Tartaglia. This last one, finished to guess that as the former has asked him to solve many third order equations, there must be a general solution, which he finally found. Then, severely hill, he asked Girolamo Cardano to cure him. The last one asked for the formula frightened that the formula be lost by his dead. Tartaglia offered it to Cardano under the promise to never revealed it. Cardano was a pretty good physicist for the time and Tartaglia recovered but having also learned all the story of the discovery, he thought that his promesse was not effective since the formula was not Tartaglia's one but the one of a dead person. So he relates the all story and the controversy with Tartaglia began.

could it be, it could hardly be characterized as brilliant, because according to Michalko (2010), brilliant ideas are productive not re-productive that is, by essence, they are not committed to a preexistent way of thinking or of doing.

Hard sciences abound of example of brilliant ideas, and mathematics perhaps more than any others, because it's the domain who necessitate the less degree of interaction with others, and the rules of the games, whichever be their complexity, could be taken over by brilliant spirits in their youth. For instance, the debasing of the notion of the infinite from metaphysics — see for instance the thomist conception of the infinity which is God singular and essential characteristic, imperfectly approachable from the finite as explain in Lindsay (1904) —, to some plural non-homogeneous mathematical notions which can be compared by Cantor (1874)[2] and more than this first demonstration, his simplification by the diagonal argument in Cantor (1890-1891), is one of those rare brillant ideas in spite of the attacks, mainly by Kronecker, which contributed to increase the psychiatric disease he suffered the most part of his life.

In the contrary, in the soft sciences, it's hard to find a consensus on what is a brilliant idea and who are the magicians. Those who rank Marx among of them are necessarily not the same than those who rank Hayek, who are not necessarily the same than those who rank Keynes because economic ideas, as other social sciences ideas, are more time and Nurture dependant.

But, on the border line, between hard and soft science, one can perhaps be convinced that, some time, there has been some idea of genius. Among them, we are persuaded that the invention of the *present value* was such an idea which would deserve to be worth to his author the recognition of a president's title of the AEA or the IEFS on a posthumous basis for the 13th century.

Don't let this paper open on your desk to rush on your Schumpeter (1954) or any other book dedicated to the history of economic analysis or economic thought as Blaug (1985), Rothbard (1995) or Pribram (1983), to quote only some, because they have no mention of him. That is not to say, that you cannot find any text in the economic literature that present the subject — for instance, you can read Goetzmann (2004) and, if you read italian, Tangheroni (2002)[3]. You can also find a too brief discussion in Rubinstein (2006).

Unfortunately the name of the man who, from scratch, invented the notion of present value, if universally known, is uncertain. From the middle of the 19th century, following the other known infamous[4] historian of mathematics Guglielmo Libri, he his known under the forged name Fibonacci — see Libri (1841) — because he refers to himself as *filius Bonacci* — son of the familly Bonacci. But, according to Devlin (2011), from the usage of the time, one could have called him Leonardo Pisano — Leonard of Pisa — or, if one follow the name under which he appears in a pisan edict of about 1240 which ascribes to him a pension of 20 lira for his involvement in the education of the people and for the consulting service in financial matters he held with the city : Leonardo Bigollo[5].

Before to reconstitute the Fibonacci's followed route to the present value, to explain why it is a more than a financial brilliant idea but also an idea which will show to be the passage of the universal thought from the adolescence to maturity, to propose a consensual simple test of its shine, we will describe the the scene under which it has been elaborated.

1 The 12th century scene

To be able to decide if an idea is brilliant or not one must start from the contemporary situation. The relevant elements which could have participate or prevent the invention of the present value are : the

[2]We will see later that Cantor has been anticipated by Nicole Oresme in the 14th century.

[3]You can read directly Tangheroni (2002) , freely and legally from `http://php.math.unifi.it/archimede/archimede/fibonacci/catalogo/catalogo.html`.

[4]Libri is known to have theft and damaged many rare books and manuscripts.

[5]In fact all of this could have been forged from ignorance, since as noted by Drozdyuk & Drozdyuk (2010), the best reference about the life of Leonardo,, in 1506, the roman empire notary Perizolo mentions Leonardo as Fibonacci.

economic situation in Tuscany, the mathematical technology and the church condemnation of the usury.

1.1 The economic situation in Tuscany in the 12th century

The 12th and 13th centuries are the centuries of crusades. Three are specific to the 12th century. As a unintended result, if Jerusalem has not been freed from arabic presence, a notable number of holy land harbours has been occupied even if a century later there will be no more occidental trading posts in the middle-east and in the Magreb. Toscan's costal cities and Venice has been among the beneficiaries of the crusades because boat transport has proved the safest means to arrive at destination. They have realised that they could fill their boats with cargo of spices[6] : ginger, pepper, saffron, cardamom, cinnamon... They discover also more rare indian luxuries : musk, camphor, ambergris and sandalwood. Of course, this commerce was only an extension of what exists since the 11th century when Bari, Trani, Brindisi and Taranto profitably established the first broad commercial links with the Levant. The trade of spices increased not only for food reasons but because it allowed an extension of the western pharmacopoeia. The wealth of those cities was guaranteed by the correlation of the absolute scarcity of the species which correlate with their prices. One must always remember that, tree centuries later, when king Manuel of Portugal authorized Vasco de Gama's voyage in 1497, it was "*in search of spices*" — see Freedman (2005).

As Gaulin & Menant (1995) clearly set it, because a long-lasting society is project seething, "*in communal Italy everybody lives on credit*". The great cities were engaged in a continual war the same against the others in the end to spread their sphere of influence and domination, assure the supply of the publics attics, pay the salaries of the state employees, build new fortifications, monuments, roads and canals. Traders borrow to get their supply and customers to buy them. The tax systems imposed by the ruling cities, forced the local nobility and ecclesiastical institutions to get in debt in an important way. And the indebtedness was not only a urban phenomenon. It also affected the countryside, by the enlargement of domains, the modernization of the agricultural equipments, the marriages, the patent consumption or still out of necessity in case of bad harvests.

Needs in liquidity were going to be supplied by the development of the commercial system set up by the Lombards[7] from the beginning of the 11th century which corresponds to the installation of the first counters in the East and in North Africa. The success of these counters was going to allow them to eliminate the storekeepers and the Jewish bankers of the oriental trade. The latter will emigrate in Cordoba then capital of Al-Andalus, the Iberic peninsula region under Arabic domination since 711.

During the 12th century, the bill of exchange[8] becomes a common mean of payment. In reality it was a borrowing to the arabs who, since the 10th century has developed the *suftadja*[9] — a thing or a commodity, which one person takes as a loan from the other on condition of their return in another place — and *hawala*[10] — a debt obligation transfer from one person to another and as debt clearing by a new similar debt — which, in itself, was a chinese borrowing which reach the arabic world through persian.

As soon as 1063, a coalition of genovese, venetian and english traders create the "*Amalfi Code*", an insurance fund which guaranteed the refund of their losses in case where their cargo sunk at sea. In 1119, the order of the Knight Templar is created to defend the holy places and protect pilgrims on the way towards Jerusalem. But very quickly, while they are supposed to respect the rules of St Benedict who, among other, impose poverty, they are transformed into an economic and financial power. First of all

[6] The world appears in the 12th century.

[7] Lombard was the name given to those who gave on security as far as, originally, this type of loan had seemed in Lombardy. In the South of France, one spokes of caorsins because of the large number of lenders settled in Cahors. In Paris, in the 11th century, a bridge was dedicated jointly to this activity and to the silversmith's trade: the famous *Pont-Au-Change.*

[8] A written order by the drawer to the drawee to pay money to the payee.

[9] See Ashtor (1972).

[10] See Schramm & Taube (2003).

warehouses, lenders to kings, to clergy, to noble persons and to traders, the establishments that they had based in Europe and in the Middle East their allowed to transport, in complete safety, big sums of gold and money(silver) and to make payments on foreign places — see Delisle (1887).

In the same way, from the end of the 10th century — Lopez & Raymond (1955) —, new forms of contracts appear: Venice creates the *colleganza*[11] which, when bilateral, binds a sedentary financier to a trader, or to a sailor, by distributing the investments for 2/3 to the first one and 1/3 to the second and shares profits by half or, when it is unilateral and becomes a contract in command — *commanda* —, because only the financier moves forward liquid assets and because the sharing comes true in 3/4 for the lessor, against 1/4 for the other partner[12]. In general, those type of contract was signed for travel. For land trades, Venice also invented the *Compania* contract, which was initially, and at the death of the father, used for apportioning the expense of ship construction between brothers, it developed into a widely used and very flexible type of business agreement. Thus, partners combined their resources, of which each partner stated his share, agreed to work for their common interest, to travel together to named or unnamed destinations, and to divide the profits in proportion to their respective investments. Liability of all partners was joint and unlimited — see Robbert (1985) and Weber (2003).

But the most important source of profit, come from the operation of change. As european operated in many regions of the world, from town yo town or from country to country, and as monies were all but stable, there was a vast field where new bankers could be free of any considerations and by-pass in any legality the moral bans raised by the Roman Catholic Church, which forbade formally the interest-bearing loan.

1.2 The church's fight against usury

Indeed, Paul's plea for an universal christendom associated with the recognition of the Bible as the sacred book which, later, after the aggregation of the four gospels and the apostles' acts, some epistles as Pauline's one... and the book of the revelations will give the New Testament, came with the ban of interest-bearing loan.

Indeed, the *Deutheronome* (20:19) is imperative on the fact that *you will not lend to interest to your brother, interest of money(silver) or interest of food, any thing which lends itself to interest*. And, in the following verse, it adds that *you can pull an interest of the foreigner, but you will not pull it of your brother, so that the Eternal, your God, blesses you in all that you will undertake in the country of which you are going to take possession*[13]. In the New Testament, the formulation of this prohibition is less explicit. In *the Parable of the bag of gold — Matthew (25:14-30) and/or Luc (19:12-27)* — Christ seems to legitimate the interests but, as noted in Thomas d'Aquin (1984[1266-1273]) (2-2, question 78), the interests question of which it is in this parable must be understood on a metaphoric plan since, in *Luc* (6: 34-35), it is specified that, if you wait for something of a loan, you will pull no gratitude as a result of which, it is necessary to give without hoping for anything. But, the legal basis in the invocation of *Luc* was very weak since, it is only an exhortation to a general and disinterested benevolence.

Moreover, according to Visser & McIntosh (1998), which lean on the *Encyclopedia Judaica*, the prohibition was of charitable nature, its violation being not regarded as a criminal offense but rather as a moral transgression[14].

And as, for a christian, even the non-believer is his brother, the interest-bearing must be banned. As so elegantly expressed by Nelson (1949) in the subtitle of his work on usure, in passing the Jewish law in

[11] de Roover (1941) speaks of *societa maris*.

[12] For the origin of the *commenda* contract one can consult Udovitch (1962) and/or Pryor (1977).

[13] In five others places the Bible does underscore this interdiction : in *Exode* (22:25), in *Book of Leviticus* (35:36), in *Ezekiel* (18:8) and (22:12) and in *Psalms* (15:5).

[14] *Isaiah* (50:1-11), *Samuel* (I, 22:2) and *Kings* (II, 4:1) and the *Elephantine papyri* written in Egypt during the 5th century B.C. suggests that creditors were exacting and implacable in their exactions.

the law of the Christ, the Christian gave up the tribal brotherhood to adhere to the universal otherhood.

Nevertheless, until the 4th century, all that can be inferred from the fathers[15] and other ecclesiastical writers is that it is contrary to mercy and humanity to demand interest from a poor and needy man — see Vermeersh (1912). The more one can find was not about a condemnation but rather about a denunciation of the moral decline and the miserliness observed and of lenders who, under the pretence to help the unfortunates, condemned them to sink even more into poverty. But, until the 9th century, in spite of a text of the Council of Elvira that has taken place in 305 or 306, and has condemned the layman, the shame reposed only on the shoulders of the clerics.

But it's only by the Nicea council, in 325, that the christian church, then roman state's religion since the conversion of Constantin the Great in 312[16], decide that the ban will take effect on the basis of *Psalm* (15:5). In 345, this ban was strengthened by the twelfth cannon of the council of Cartage and then, in 789, by the 36th cannon of the council of Aix. This time, even the laymen were under the blow of the ban.

As underline by Moehlman (1934), Charlemagne and the councils of the 9th century applied the regulation to the laity, but in that time, it was only reprehensible to lend in interest. And for proof a the paranoid attitude of the church, the concil of Celchyth — 816 A.D. — laid down the rule thaht the land of the Church should not be charged for more that the term of a single life which is, de facto, what actually meant that the higher authorities of the Church recognized that to lean was an usual operation. Amazingly, Kopf (1927) quotes with delectation, the case the abbot of St. Denis who, in 1308, arranged with the archbishop of Cologne to pay a life annuity of 400 livres to the archbishop in consideration of the 2,400 livres paid to the monastery. In event of death during the two years of the term of the annuity, 1000 livres were to be re-paid to the heirs of the archbishop.

According to Piron (2005), around the middle of the 12th century, the usure question was considered as secondary. In fact, after the ruggedness of the second council of Nicea, it seems urgent for many writers to conciliate with the practice of usury. For instance, in the twelve canon of his 158th epistle, St Basil states that a usurer may even be admitted to orders, provided he gives his acquired wealth to the poor and abstains for the future from the pursuit of gain ; Gregory of Nyssa confirms that usury, unlike theft, the desecration of tombs, and sacrilege, is allowed to pass unpunished, although among the things forbidden by Scripture. Moreover, a candidate at ordination should never be asked whether or no he has been guilty of this type of practice. In one of his epistles — the epistle 6.24 —, Sidonius Apollinaris appears to imply that no blame should be attached to lending money at the legal rate of interest, and that even a bishop might be a creditor on those terms. Even Gregory the Great in his epistle (ix. 38) seems to shew that he did not regard the payment of interest for money advanced by one layman to another as unlawful.

But, forty years later, that is in the second part of the 12th century, the council of Tour — 1163 — and of Lateran III — 1179 — canons gathered under the authority of the pope Alexander III have began to set the scene[17], reinforcing the decision of the Lateran II — 1139 — which, throughout its canon 13, for the first time, condemns usurers to excommunication and deprives them of Christian burial. For instance, Lateran III decreed :

> *Since in almost every place the crime of usury has become so prevalent that many persons give up all other business an become usurers, as if it were permitted, regarding not its prohibition in both testaments, we ordain that manifest usurers shall not be admitted to communion, nor, if they die in their sin, receive christian burial, and that no priest shall accept their alms.*

This was not an idle threat. The numerous predicator sermons which get through us and has been written by Thomas of Cobham, Jacques de Vitry and others, testify that the catholic church planed to

[15]For the occidental church, the tradition tradition account four fathers : Ambrosius, Augustine, Gregory and Hieronymus.

[16]He will be baptized only on his deathbed into 337.

[17]Even if, before the 13th century, council after council, usury has been condemn : In the canon 3 of the council of Carthage — 348 —, in the canon 4 in Laodicea — 343-381 —, in the council of Mainz — 813 –, in the council of Rheims — 813 —, the one of Chalons — 813 —, the Council of Aix — 816.

foster the fear of eternal damnation in the mind of christian usurers. And this fear was so pregnant, that some rich usurers of the 12th century engage themselves on the restitution way on their dead bed. One must notice that Le Goff (1975) cite no example of a court decision according to usury. To take the mesure of the extend of the application of the canon laws, one must consult Helmolz (1986), who tried to expose and clarify the judgment of the english courts for which one disposes of very partial information which stretched out off the 12th century.

Obligate to shed an interest payoff, whichever be it small or high, was unlawful and liable of a strict sanction : the excommunication of the sinner. This entailed not only exclusion of the sacraments but also from the normal company of the christians and as Helmolz put it, this was "*a considerable penalty under medieval conditions*". Convinced usurers were obliged to restitute the interest to their victims or, if this condition was impossible to fill due to the decease of the borrower, to affect it to charitable uses. But, Apart from the well know huge and first acknowledged restitution operated by the very efficient genovese trader Blancardo on his dead bed in 1178 — see Nelson (1949) —, the acknowledge restitutions were frequent as in the case of Iacopo Angiolieri, Folco Portinari — the father od Dante's Béatrice — Federico Rimpretto, Gandulfus de Arcellis and later Jakemon of Louchart (1285)[18]. In the same way, in 1175, the wealthy merchant of Lyon named Vaudès eschew all his terrestrial properties and decide to live in a strict imitation of Christ, before to be excommunicated for heresy in the concile of Verona in 1184.

Even if offenders often ignored citations, and disobeyed decrees, prosecutions were undertaken and carried forward widely enough in such a way that the canon law was all but dead letters.

In this times, usury was interest of any kind, and the canon merely forbade the clergy to lend money on interest above 12.7 per year. This is the key of what was really intended by the sin of usury which was primary facies a sin against justice as revealed by Noonan (1957). The lenders, whom the Church had in its sight, constituted a group clearly identified, considered as notorious public usurers, typically jews and lombards, which loaded interest rates of 35%. According to Bédarride (1861), often lombards and caorsins used to fix monthly interest rates of 10 %. Up to this point, it is mandatory to remark that, because of the ban to give to interest loans, liquid assets were rare and what in front of an important demand pulled by bigger and bigger needs, their prize could only burn up.

In what concern merchant bankers, who provided modest interest rates and who paid modest interest on deposits, even if from time to time they were looked with an malevolent eyes, they were not considered as sinners. Even more, at the end of the 12th century, laity and trading were no more an exclusion clause to the access to the communion of the saints, as gives evidence the canonization of Homebon of Cremona — see Vauchez (2003)[19],[20].

From the fall of the roman empire until the 12th century, it was not too difficult to respect the prohibition, since the frugality of the needs restricted the requests for loans of clothes, of food and of seeds. But suddenly, the loans become the big problem of the trades. It's, no more and no less, the question of the allocation of scarce resources on new uses. Now, if one accepts the idea that on that subject the people are rather rational in all times, to give it is necessary to hope for a yield.

Nevertheless, from many indirect sources one knows that, despite its decreed interdiction, the church permitted the taking of interest because, as noted by de Roover (1967), "*contrary to what many believed, bankers did not simply disregard the usury doctrine, but they made an effort to comply*". As long as it was possible to justify it as the income of a capital invested in production and concealed in profit, there was no objection to it. If one lend a given sum of money, it was considered as against the christian credo to claim for a remuneration. But the same amount of money could be invested in business and, in full legality, generate a huge profit.

[18] All those names are cited by Nelson (1949).

[19] There is a strong contrast with Guidon d'Anderlecht, canonized in 1112 because he was able to resist to the the temptation to make some trade on the Zenne river — Belgium.

[20] Until the canonization of Homebon, outside the clergy only noble warriors have been canonized as William of Gellone — 1066 —, the english king Edward the Confessor — 1161 —, king Olf II Haraldsson of Norway in 1164...

It has been noticed elsewhere — see Rubin (2010) —, that what distinguished the christian from the muslim interest ban was the fact that, happily, for the economic development of the formers, civil authorities never tried to endorse it. One can attribute this state of facts to the paranoiac dilemma in which, by nature, the strict application of the catholic dogma dives not only the church, but all the parts of the medieval society.

Following Piron (2005), one can observe that prohibiting usure created the outstanding fact that a religious imperative contradict a moral obligation conceived as a natural right. Indeed, what ever be its justification, as the time robbery[21] which is an argument hammered in Peter Cantor's sermons at the end of the 12th century — see Baldwin (1970) or Le Goff (1975), the interdiction to remunerate the services done by the lender, directly contradicts the social morality rule which inflicts to thanks ones benefactor and to give proofs of ones thankfulness. In short, the usury ban obstructs the reciprocity law. By a strange twist of the appeal argument, the catholic glossators will subvert the strongness of the usury ban.

There would be a natural obligation resulting from undiluted movements of the spirit — animus — by which every creature is moved to do good to his benefactor. And this obligation would be more effective than the naked pact. This is the notion of *antidora*[22]. This has as consequence that the sin doesn't reside in the interest per se but in the primary hope to receive an interest. The lender must be only submitted to the moral obligation to thanks the lender but this last one has nothing to hope.

The emergence of this scholastic argument which, practically, will prevent any effective fight against usury which, little by little will turn out to be a lost fight, since the technical progress implied by the always increasing efficiency of the resources allocation, which by itself will engender the trendy fall of the rate of profits, on which Marx so much insisted in the Capital — see Marx (1967[1884]) — has for origin the intellectual event which will, from the end of 11th century, re-establish the economic and social dynamism of the west: the rediscovery of the *Digest* of Justinien[23].

The redaction of the Digest under the direction of the jurist Tribonian is a compilation of the civil roman laws redacted between 529 to 534. As the team of jurists gathered by Tribonian were authorized to edit the laws, one cannot say if they have been largely amended in comparison with the original ones and one cannot say how far it was effective in the two empires, with the oriental accruing problem that it has been redacted in latin when the common language was the greek. The only thing one can tell is that, after that, in the West, Pope Gregory the Great last cited it in a letter of 603. Later sources remain silent for nearly 500 years. In the Est, the bizantine empire continued to used it, after many updates preformed to take in account the evolution of the civil society. It give birth to the *Ecloga* code of 740, enacted by Leo the Isaurian, to the *Proheiron* in 879, enacted by Basil the Macedonian and to the Basilika in the late 9th century which began under the auspices of Basil the Macedonian and was finished by his son Leo the Wise. According to Mueller (1990), in the West, its long absence is due to a mixing between the general decline of learning and the fact that he couldn't speak to the rare early medieval reader since it was the expression of a far more sophisticated urban society compared with the one which succeeded it.

One must wait 1076 for a citation of the digest in a document issued in Marturi, a small city of Tuscany. Then in 1093 or 1094, Yvo of Chartres included numerous excerpts from the books 1 to 24 of the Digest into his collection of canon laws. Many scholars have attributed the rediscovery of the digest to the importation a the unique complete version which remains of the Digests : the *Codex Florentinus* once the possession of Pisa[24], which has been taken by Florence after that Pisa has felt under the florentian domination[25]. According to the main legendary tale of it's founding by the pisans, it should have been

[21] Time belongs only to God.

[22] According to Piron (2005), the Middle-Ages standard written form for *antidory* is *antidota*. That is to say antidote, *dosis* having simultaneously the sense of gift and poison ether in greek or in german.

[23] Or, *Pandecte* if one prefers greek to latin.

[24] It has been given to the city by Burgondio of Pisa, after his return from a sojourn from Constantinople in 1135-1140. It has been considered as the most precious manuscript in the world after the Bible.

[25] Property of the florentian republic since 1406, it is kept at the *Biblioteca Medicea Laurenziana* since 1786.

bring back as a war seizure after the fall of Amalfi in 1134 or 1125. But, as clearly demonstrated by Mueller (1990), as soon as 1112, Irnerius, one of the firsts glossators[26] of the Digests began to teach roman law on the basis of an other defective and strangely arranged bolognese version of the Digests. The four doctors Martinus, Bulgarus, Hugo and Jacobus made also reference to this version which certainly predates the *Codex florentinus*.

But, whatsoever be the complex process of its recovery, a large part of the bolognese jurists were all in a moment fascinated by this piece of writing which was showing that an other organisation of the ties which wreathes the social network than the one proposed by the catholic church has been in use in a not so far past. Of course, many of the justinian laws should have been rethought and rewritten to fit in the Middle-Age society, but they offer some answers to the question that the growing italian towns elicit.

It must be noticed that if feudalism, that is, according to Ganshof (1952[1944]), a set of reciprocal legal and military obligations among the warrior nobility, extended by Bloch (1961[1944]) to encompass the three estates — the nobility, the clerics and the peasantry —, expands itself in southern and western europe, in germany and in eastern europe, the great byzantine empire, and many important italian towns as Venice, Genoa, Milan, Florence, Pisa, Amalfi and, first of all, Rome never once abjured the allegiance to the roman law. More than that, as underlined by Morris (1916), every town and city, that became a bishop seat became simultaneously the refuge of the roman spirit and, paradoxically according to the church position on many subjects, a centre for the propagation of the principes of human liberty. And this applies as well for the Italian cities as for the big rhenans ones or those of the Baltic.

But, even if in place since a long time, what happen in the end of the 12th and in the beginning of the 13th century is a fight between what one can call the ancients and the moderns. In the end, in part because of the reluctance of the civil and religious authorities, which oversaw the old universities to give to the civil law an equal place to the one of the canon law, the University of Bologna was victim of a secession which gave birth to the University of Padua. A few time later, Paris' University, after its refusal that civil law be taught behind its walls, saw the birth of the University of Orléans. But all this append in the articulation of the first et the second quarter of the 12th century.

Be that as it may, the interest-bearing loan found a legal justification in the digests and in a world where the power of the church was perpetually in conflict with the power of the many kings and emperors it must composed with, the church did not have more than a solution: to stiffen his position. It is what happen in 1215, in the fourth council of Lateran, conducted by Innocent III under the pressure of the newly created *order of friars minor* by St Francis of Assisi — 1210 — and the *order of preacher* more commonly known as the *Dominican order* from the first name of his creator Dominic de Guzmán — 1215.

Even if one can find seven things hatred by the lord in the *Book of Proverbs*, the idea of a classification of the most important sins is attributed to Evagrius Ponticus, a christian monk and ascetic who lived in the fourth century — see Bloomfield (1952). As an ascetic, he naturally classify gluttony as the first of all sins, prostitution and fornication as the second and avarice as the third. Two centuries later, the pope Gregory I the great revised marginally the list, essentially inverting the two first sins. But, during the fourth council of Lateran, the structure was definitively fixed, in the order of pride, miserliness, wrath, envy, lust, lazyness and greed. In the same council, the personal confession was also introduced in a periodic setting — at last, one time a year.

The elevation in the rank of the deadly sins of miserliness was clearly at attack against usury since the opposed virtue — generosity — encompass the fact of giving without having expectation of the other person, when miserliness want is fair share or a little bit more. But, lazyness also sends back to usure as far as the interest cannot be identified with labor because that one requires some rest, while investment in securities is an operation which metaphorically never sleeps. And if confession has been made mandatory it was to enforce repentance which is the act of reconciliation of reconciliation in the body of Christ because, as express in Paul corinthian's epistle — *1 Cor 1:26-28 — if one member suffers,*

[26] A gloss is a linguistic comment added in the margins or between the lines of a text, to explain a foreign, a dialectal or a rare word or term.

all the members suffer with it.

From the economic point of view, this stiffening of the church position could be understood if one looks at the church as a corporation aiming at monopolising the religions market by regulating social norms — sin and redemption —, eliminating competition and regulating all forms of social exchange from usury, to marriage through trades. This is the standard neoclassical point of view defended by Anderson et al. (1992), Ekelund et al. (1989), Ekelund et al. (1992) and collected in Ekelund et al. (1996). The main idea is that the christian church was maximising its incomes from rent-seeking.

It must be acknowledged that, in the 10th-12th century, catholic church looks like a monopoly fighting, on one hand, the entry of new competitors and, on the other hand, the established powers. First off all, there was the heresies, as the cathars' one, which constituted a class in his own, the evangelic dissenters as the waldensians — those which followed Vaudes —, or the Humiliati, the pantheist dissenters as the Amalricians — those who followed the teaching of Amalric of Bena — which taught that god was the formal principle of all things, and that every single person was as much God as was Christ, the preachers and iconoclasts which were isolated chiefs as Peter de Bruys, Henri of Lausanne[27], Tanchelm... — see Leff (1999[1967]). In a certain way, muslims and jews were aggregated as heretics, what explains that crusade against the first ones and the violence against the second were organized.

In what concern the fighting with the terrestrial powers, first of all there was the conflict of investitures. In Germany, under the Ottonians, the temporal power had got used to appointing the bishops to its please. Therefore, a large number of bishop's palaces were kept by corrupt men totally devoted to the emperor. With the Gregorian reform, which is going to be implemented from the 11th century, the church wanted to reconquer the investitures. This could only lead to a conflict with the emperor. This conflict last until 1122, were the Worms' concordat ruled for the free election of the bishops. At that time, the conflict originated also in France and England has ceased since a long time, essentially because of the network of monasteries which had propagated the roman influence.

The conflict could only be amplified by the promulgation in 754 by Pepin the Short who, having made a commitment to give up to the pope the lands whom he has conquered on the Lombards, promulgate the *Constantine's donatio* which would have given, in 335, to the pope Sylvester I the superiority onto Eastern Churches and imperium on the West[28]. From this day on, the christian church was also a temporal power and it was obliged to adapt its politics to its dual nature : temporal and spiritual[29].

1.3 The mathematical technology

Unfortunately, whatever be the religious or civil legislations regulating the interest loans, the mathematical technology necessary to calculate compound interest was for millennium more than deficient. For instance, Middle Age traders inherited the roman numeral system, which was highly limited when performing the most simple operations.

In fact, this assertion is not completely true. According to Turner (1951), the roman numerals which are, by nature, additives[30], were even more easier to add or to subtract than the hindu-arabic ones. For instance, subtracting or adding 126 — CXXVI — to 378 — CCCLXXVIII — is a very simple calculation which is perform without knowing any table as shown here after[31] :

[27]Respetively, their sects was named Petrobrusians and Henrisians.

[28]In 1440, the humanist Lorenzo Valla demonstrated that it was about a forgery.

[29]To be more precise, Constantin used christianity as a mean of governance doting the local churches and bishoprics with huges incomes. Its not a pure hazard, if he decides not to build a cathedral on the tomb of St Peter but a basilica since by this simple denomination of the church was acknowledging the right for the bishop of Rome to dispense justice, since, contrary to the cathedral which is the seat of the bishop, the basilica, in the Constantin's time, was the place on the forum where judicial activities were led. Unfortunately, banking and commercial markets were also common in the basilicas.

[30]That is to say : 1977 or MCMLXXVII is, literally written, in roman numerals notation as $1 \times 1000 + 9 \times 100 + 7 \times 10 + 1 \times 7$.

[31]X = VV, L = XXXXX and D = CCCCC.

	CCC̸	LX̸X̸	V̸III	(378)
-	C	XX	VI	(126)
	CC	L	II	(252)

	C	X		
	CCC	LXX	VIII	(378)
+	C	XX	VI	(126)
	D	(C)	(X)VI	(504)

At least, since Cajori (2007[1898]), the idea that multiplication and division were truly hard problems if carried out with roman numeral has diffused among scholars. On that subject, reinforced by the fact that no multiplicative roman calculus has never been discovered in the document inherited from roman or middle-age time, the common point of view was that those operations were performed through the abacus. But, as shown by Anderson (1956), it's a profound mistake[32]. For instance, in the case of the multiplication of 17 — XVII — by 60 — CL —, one has :

	XVII					
×	LX					
	D	CCL	L	L		
+		C	L		X	X
	D	CCCL	C	L	X	X
=	D	CCCCC			X	X
=	D	D			X	X
=	M				X	X

17
60
00
102
1020

One can observe that the main difference between the two operating mode is that with roman numeral the operations are carried out from left to right contrary to the ones carried out with hindu-arabic numerals.

In what concern the division, there are no essential differences between the two types of numerals. For instance, to compute 209 ÷ 11 that is — CCIX÷XI —, one has

C	C	I	X	X	I
C	X			X	
	LXXXX	I	X		IX
	LXXXX	I	X		

20	9	11
11		19
9	9	
	0	

But, as is not instantly apparent, division is in fact simpler with roman numerals than with hindu-arabic ones because, it entitles the possibility to not find the exact number of time the divisor will go in the dividend at each step[33]. Anderson (1956) also shows how to calculate powers or how to extract square roots. This experimental archaeological explanation shows that, contrary to the general conviction, it could be conjectured that it is not because of its intrinsic complexity that roman numerals were finally derelict. Unfortunately, the mathematical culture of the mid-twentieth century scholar was so greatly disconnected with that of his Middle-Age commensal, that it is certainly an event with zero probability that anybody could have been able to compute easily a division, to the exception of the trivial ones. This could be exemplified by the Claude Shannon 1941 construction of a four operations calculator using roman numerals — see Shannon (1993).

However, one knows that from roman time, peoples were taught to count on their fingers, but even for somebody who mastered pretty well the technics, it was not very efficient. And even after the creation of

[32] In his appendix, Anderson (1956) take a brag pleasure to carry out the multiplication of DCCXXIII by CCCLXIV reputed by Cajori as impossible to compute with roman numerals.

[33] One must note that, in all those cases, there is a flavor a heresy in the notation used conjointly with roman numeral since the sign + appears for the first time in a book by Hodder in 1672, the sign × is a joint product of the algorithm by Napier in 1618 and ÷ is due to Rahn in 1659. On those subjects, one can consult Smith (1958) and Cajori (1909).

the schools under the impetus given by Charlemagne and Alcuin, the fight against paganism was a factor of elimination of any mathematical education because of the paganist flavor of mathematics. Only in the school of the York cathedral, elements of mathematics were taught.

One knows also, that even if developed in China 3000 years BC, the Middle-Age peoples have inherited of some counting tables collectively named *abacus* from the romans. Unfortunately, this method has a heavy drawback : it keeps no trace of the achieved operations, the final result being destructed by the beginning of a new operation.

The defectiveness of the roman numeral system was largely noticed by the very few who were able to think about it and to try to find a solution. The first to attempt really to make the situation evolve was Gerbert d'Aurillac[34], the son of a poor shepherd, who writes his primary works when he was head of the Rheims cathedral school — 972-989 — and who finally will be elected Pope under the name of Sylvestre II in 999. Gerbert was himself a mathematician who has extensively studied arabic mathematics throughout his youth travel in the iberic peninsula from Al-Andalous — mainly Cordoba — to Catalugna. As he discovered some interesting document, he favored the study of Boethius, one of the rare romans who studied mathematics in the 5th century[35]. But, as Boethius was himself interested by the work of the surveyor, the mathematics studied had a more pronounced flavor of geometry than the arithmetics needed to keep the accounts.

Gerbert and his pupil Bernelin, who has written a book dedicated to the finding of his master — see Bernelin (2011[999]) —, envision an abacus where hindu-arabic roman numerals take the place of the roman ones. Unfortunately, Sylvestre II mortal remains undergo the costs of the try. Because of his efforts in the transmission of the arabic science and mainly because, attached to the revelation of the hindu-arabic positional system of numerical notation which was, in that time relatively new, attached with this new system was the zero — initially *zephyrum* — who was no more than the arabic *shifr*, which, itself, was no more than the indian *sunya* which stand for void[36], a veil of sorcery stayed associated with his name, until his exhumation in 1648 to verify if he was still inhabited by all the devils of inferno.

One can conjecture that there was a disenchantment between Middle-Age thinkers and roman numerals. If for scholars as Murray (1991), the original sin of roman numerals was to embalm the primitive principles of addition and substraction in such a way to block the entry, into notation, of the multiplication and the division, a point of view which is dismissed by the former paragraphs, King (2001) sees a conjunction of many practical different causes. For example, the roman numeral were not fitted to accommodate great numbers because the letters used to construct a number were limited. As they used X for ten, C for one hundred and M for a thousand, they were also obliged to write a horizontal line through or above the numerals to raised the value of the number by a factor of 10 or of 1000. King (2001) displays a tentative by a certain Adriaen vander Gucht, realized in 1569, to construct a table of power of 10 from 1 to 29. But as one can observe, it couldn't be used because of the ambiguity here attached.

[34] On Gerbert as a teacher, one can consult Darlington (1947).

[35] Until the beginning of the XXth century, Boethius has been credited to the first usage of the hindu-arabic numerals in Occident. But, Smith & Karpinski (1911) have erased this credit, showing, by an erudite demonstration, that a man of his inquiring mind couldn't have let aside the least quantum of information that spreads out in the mediterranean world through commercial relations, but, as in his time, the zero has not yet been incorporated as a place holder with the other numbers, the arabic ciphers where not an improvement in the calculation effort in comparison with roman numeral. So there where no reason for Boethius to use them. In what concerns his *Geometry*, one of the three works which are attributed to Boethius, where one can found the complete hindu-arabic numerals including the zero, Smith & Karpinski (1911) have demonstrated that it has been belatedly falsified since, if Boethius have known such a system, he would have used it mainly in his *Arithmetics*, that none of his disciples and successor never used it, and that the falsification of text was under complaint even in his own time.

[36] On the transmission to Gerbert of the hindu-arabic numeral, one can consult Zuccato (2005).

Number	Representation				
I	$10^0 = 1$	10^6	10^{12}	10^{18}	10^{24}
X	10^1	10^7	10^{13}	10^{19}	10^{25}
C	10^2	10^8	10^{14}	10^{20}	10^{26}
M	10^3	10^9	10^{15}	10^{21}	10^{27}
XM	10^4	10^{10}	10^{16}	10^{22}	10^{28}
CM	10^5	10^{11}	10^{17}	10^{23}	10^{29}

As shown by the anecdote retrieved from Suetonius' *Life of Twelve Caesars* by Kaplan (2000), the roman numerals limited number of signs was a true deficiency which could conduct to many legal resorts. When dying, Livie, August's wife and certainly one of the most powerful women of the ancient times, decided to bequest to Galba 50000000 sesterces, that is to say, in roman numeral notations $|\overline{\mathrm{D}}|$. But, the son of Livie, the emperor Tiberius, who doesn't like Galba, insisted that $|\overline{\mathrm{D}}|$ be read as 500000 because *quia notata non perscripta erat summa*[37].

This may seems a venal sin of roman numerals but, as shown by Murray (1991), as time goes by, with the help of the need for large numbers, roman numerals became truly impractical. For instance, Murray reports a 1649 selling of the english crown lands for a amount of 1 423 710 pounds, 18 shillings and 6 pence which have been noted by the exchequer

		ƚ				
l⁄	C	M	C	ñ	s⁄	d⁄
M	iiij	xxiij	vij	x	xviij	iij

Even for smaller numbers, during the 15th century the roman numerals became inappropriate as a numbering system as one can realize in looking to the 24 hour clock-face situated on the inside of the western wall of the duomo Santa Maria del Fiore in Florence.

A second problem was that scribes had introduced some variations which were very difficult to decipher, as in some french manuscripts written between the 13th and the 15th century as :

$$\begin{matrix}\mathrm{XX}\\ \bullet\\ \mathrm{IIII}\end{matrix} = 81, \quad \mathrm{IX}^{\mathrm{XX}}\mathrm{III} = 183, \quad \mathrm{VII}^{\mathrm{XX}}\mathrm{VI} = 156$$

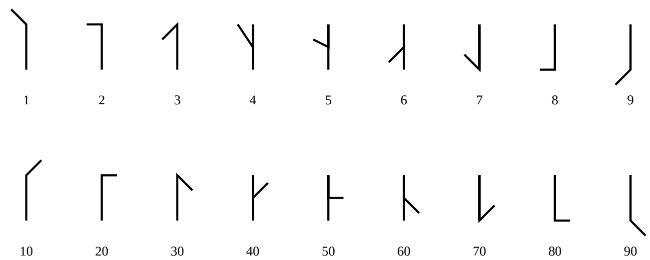
as is shown in Lehmann (1936). An other point which renders caduc the usage of the roman numeral was that, since the introduction of the hindu-arabic numerals, peoples used to mixed them with the use of roman numeral. For instance Bergner (1905) — cited by King (2001) — brings out some german inscriptions such as :

mcccc8	1408	stamp of an Augsburg religious dignitary
1• 4 • Lxiii	1463	on a gravestone from Salzburg
14XCIIII	1494	on the altar of St. Othmar in Naumburg
1 • V^{C} • V	1505	on a bell in Keila near Ziegenrück
1 • V^{C} • 6	1506	on a bell in Neustadt a. O.
15X5	1515	in Lauffen near Rottweill
MD.25	1525	in the Schlosskirche in Chemnitz

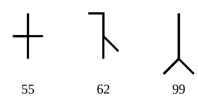
[37] Because the sum was in notation, not written in full.

And this degenerated usage was not the only appanage of germans, since even the great french humanist Guillaume Budé use it to note the year 1534 by M.5.34.

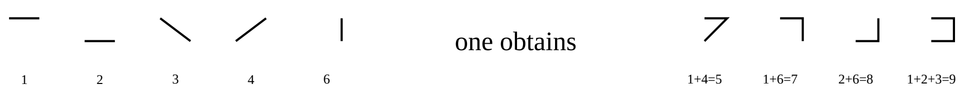
One couldn't say if this out of favoring has been the origin of the introduction of some to day forgotten numeral, but as shown by King (2001), during the 10th century, John of Basingstoke, who was one of the first in england to master greek, is reputed to have introduced some new ciphers deemed of greek origin[38]. Those ciphers were very simple but they were not able to note numbers greater than 99 as is shown in the following table.



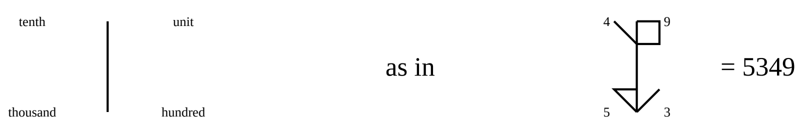
With such ciphers one can write very easily numbers as



Even if of limited range, this type of ciphers pleased at least the cistercian monks who designed some clever extensions. As shown by King (2001), cyphers of that type flourished in Europa. During the 13th century north of France, a clever vertical version of this type of cyphers, of whom design was particularly well fitted for arithmetic operations, reaches the academic mediums. Its particular structure not only permits operation up to 9999 but, thanks to its additive structure, made easy the four standard operations. From



Each number is associated with a decimal value — unit, tenth, hundred, thousand — according to its position and symmetry along a vertical bar — *i.e.* :



Even if no one has ever tried to compute with this ciphers, King (2001) has shown that they were perfectly apt to realize the four operations. But there was a major disfunction : it was impossible to define an algebra since if we define addition or multiplication as the internal operation there are numbers not representable with this numeral systems[39]. Be that as it may, this type of cyphers hardly has been

[38] As a friend of Robert Grosseteste, chancellor of the University of Oxford and archdeacon of Leicester and later bishop of Lincoln, John of Basingstoke occupied a central position in the intellectual medium of the tenth century.

[39] In 1953, then at *Bell Labs*, Claude Shannon constructed a mechanical calculator which computed in Roman numeral so as to demonstrate that, even if difficult, it was possible to compute this way.

of any used outside one 14th century astrolabe of Berselius and it's use for marking volumes on wine-barrels[40].

Before the end of the Middle-age, not only the numeral notation was deficient, so it was, at least in Europa, for the technology of the operations. Multiplication and division was hard tasks operated on hands or with an abacus.

Of course, not later than the 5th century, indian mathematicians imagined a positional system with a peculiar use of the zero since both, the positional system and the zero, were the basis of the sexagesimal system of the babylonian[41]. What make peculiar the indian numeral system is that, contrary to the 3th century b.c. babylonian numeral system, the zero was not just a mark to indicate a void position, but a true number.

The transmission through the arab mathematics to the west could only succeed in a civilization clash because, when the occident discovered the zero, it was under the monopoly of the christian doctrine which, at that time was completely devoted to the aristotelician philosophy for which the void associated with the zero was truly a non-concept.

First of all, as underlined by Seife (2000), zero brake the rubber band property of the integer numbers which comes throughout multiplication and associativity. That is to say that, for every number, with the exception of zero, the multiplication by any integer change the scale, as if the rubber band expanded. Secondly, if one adds two numbers and multiply the resulting one by any other number, one obtains the same result as if one has multiply the two primitive numbers by the same number and then adds the to resulting numbers. But zero obeys its own rules : any number multiplied by zero shrinks to zero and any number adds to zero stays unaffected. Thirdly, as the pythagorician's doctrine ascertained that every things that make sense in the universe had to be related to a neat proportion, zero couldn't fit in this doctrine as any number divided by zero is infinite, to the notable exception of zero which, as is universally known, is undetermined.

So, to the notable exception of pythagoricians, if greek mathematics could accepts irrational and negatives numbers, avare of the sumerian sexagesimal numeral system which, as one has seen earlier, has a peculiar zero, they could not accept zero as a number on philosophical basis, not by ignorance.

And then, Zeno of Elea sets out his famous paradox of Achilles and the tortoise. If Achilles raced the tortoise that has a head start, even if he runs two time speeder than the lumbering tortoise, he will never catch up with it since each time he made half the distance between his position and the one of the tortoise that last one has increase the distance that Achilles must make[42]. Because of the lack of the concept — greeks have no word to name it —, it was impossible to cope with the paradox. Zeno conclusion was contrary to physical experiment : motion was impossible. Every thing is one and changeless[43]. Here enter Aristote, who in front of Zeno paradox, declares that there is no need of infinity which could exist only in the mind of mathematicians without actual support. There was no infinity and no void, the earth was situated in the center of the universe surrounded by spheres in a pythagorean harmony. But, as there were no infinity, the number of spheres was necessary finite, the last of them being the celestial vault. There was no such thing as beyond the celestial vault and the universe was contained in a nutshell.

The reason why this conception of the universe lasts so long is that it incorporated a proof of god

[40]In the middle of the 16th century, Gerolamo Cardano proposed his own version of the ciphers in his *De subtilitate Libre XXI* of 1550 — see Cardano (1999[1666]).

[41]One may wonder why babylonians used a sexagesimal system. One possible explanation is that with such a system 1/2, 1/3 and 1/6 are integers which simplified astronomical operations.

[42]The solution will emerge from the development of mathematical analysis, with the introduction of the notion of limit by the indian mathematician Mādhava of Sañgamāgrama in the 15th century who was in all probability not aware of the problem, two centuries before that Cauchy strictly develops the analysis of the subject on the basis of the series convergence initialized by d'Alembert.

[43]More exactly, Zeno was a member of the eleatic school founded by Parmenides and his paradox was designed to support Parmenides arguments.

existence. As there was movements[44], something must be causing the motion and, by necessity, it could be only the prime mover — *i.e.* : god. As expressed by Seife (2000), as wrong as it is, the aristotelian was so successful that for a millennium it eclipsed all opposing views.

It causes many problems, the most part of them being linked to the calendar since it was impossible to define a year zero. As the catholic church endorsed that philosophy, one should wait 1277 to see Étienne Tempier, bishop of Paris, condemn all doctrines contradicting god omnipotence which was the case for the aristotelian physics — see Piché (1999). All of a sudden, void was allowed since an omnipotent deity has not to follow any consequences of a human philosophy.

Unfortunately, Tempier condemnation was not the final blow of the aristotelician theory. The church will stay clung to it for still some centuries, and even will reinvigorate this theory when attacked at the reform time.

In what concern, the operation of compound interest, we know that, at least since the bronze age Sumer, it was an operation which belong to the apparatus of the mathematical scribes As shown by Lewy (1947), one possess a lot of evidence on that matter, as the MLC 2078 cuneiform tablet edited by Sachs, Goetze & Neugebauer (2000[1945]) , the VAT 8528 and the Strasbourg 366, ll. 1-5 cuneiform tablets edited by Thureau-Dangin (1938) which all demonstrated clearly that the operation was mastered by the scribes.

Unfortunately, contrary to what as been assumed naively in a first time — see Roscher (1878) or even in a recent time — see Foster (1995) —, the ancient interests rates where not market prices, they were administered prices — according to the will of temples, kings or even tradition. This could explain why they remains relatively stable over the centuries even if declining in the overall times. Recently, Hudson (2000) has proposed an explanation of this well known phenomenon bases on the simplicity of the calculations.

First of all, sumerians were using a sexagesimal numeral system. As men begin by finger counting it is always a question to understand why a non decimal system has been in use in some place ? The answer is astonishing simple : 10 has only four divisors, 1, 2, 5 an 10 while a duodecimal system — a base 12 — has six of it, 1, 2, 3, 4, 6 and 12. So if sumerians have chosen a sexagesimal system is only because 60 has 12 divisors, 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30 and 60. Which any base numeral system of number it is not too hard to realise the three first operations, addition, substraction and division.

But in what concern divisions, it was an other problem. For instance in a sexagesimal numeral system, it is possible to expressed exactly only regular number that is number that can be prime factorized only with 2, 3 or 5. For instance, 1/12 of 60 is simply 5. But as one have been taught when one has studied the clock numbering inherited from thos times, one 1/8 of 60 is 7:30[45].

2 Liber Abaci

For those which have open *Liber Abaci*, it is immediately apparent that such a book could not have been written in a world also bubbling than the one of the west in its Middle-Age. It's a vernacular book that has necessitated some long years of gestation. One can praise the premonition William, Fibonacci's father, who, while a public official of Pisa in Bugia[46], advised him to attend the course of the Arabic mathematicians. The young Leonardo Pisano[47] was so pleased to this first lessons that he crossed throughout all the

[44] As the earth was at the center of the world, it couldn't move. So, the rotation of the celestial on itself should be explained.

[45] Seven minutes and thirty seconds.

[46] Bugia, Béjaïa, Vgayet or Bgayet, according to which language you use is now an Algeria town.

[47] In any case, was Fibonacci known under this name in his life time. In the tradition of that time, he would have been known as Leonardo of Pisa. But, in the introduction of *Liber Abbaci*, he refers to himself as *filius Bonacci*, a name which was not the name of his father William — one knows also the name of a brother Bonaccingus, but nothing. Perhaps, was he proud to belong to the family of his mother. In 1838, the historian Guillaume Libri take the freedom to contract *filius Bonacci* in Fibonacci. Occasionally, he refers to himself as *Bigollo* which, in tuscan dialect signifies traveler and in others italian dialects blockhead. It is under this

Mediterranean Sea to find whoever was learned in it from Egypt, to Syria, Greece, Sicily and Provence. In other circumstances the book would certainly not has been written or should has not incorporated so many worked examples. Then a few years before his thirty — he was born in 1175 —, in 1202, he came back to Pisa, edit his work, certainly open a school, and in the twenty years that followed began to practice financial advice. The success of his *Liber Abbaci*, *Practica Geometria* was so great that he realized a second edition publish in his living in 1228 after the dubbing he obtains when the emperor of the Holy German Empire Frederick II summoned him and made him have some tests of which he got out successfully.

Up to now, all copies of the 1202 edition have disappeared. For what concerns the 1228 edition only fourteen copies subsist nowadays but only three, located in Italia, are complete or almost so and seven are mere fragments. It should wait 800 years to be translated in English — see Fibonacci & Sigler (2003[1202]).

What make Fibonnaci book so peculiar to his temporaries ? The answer is astonishingly simple : it provided an original sum of knowledge linked to the economic of trade which, up to that innovation hardly at disposal even in the old technology of the roman numerals and, in a time where illiteracy was the rule, were, at the end, easy to memorize and manage.

Not only it was the first occidental exposition of the hindu-arabic numerals which diffuses outside the friaries, but it gives also the algorithm to compute the four operations and explains how to apply them to problems essentially linked to the mathematics of trade, freeing the accountants of any counting boards, contrary to what announced the title of the book since, until then, they were mandatory to compute any operation[48]. As such, he was immediately adopted by merchants who, unexpectedly, could, all of a sudden, acquire, in a time were knowledge was nearly a church monopoly, a modern technology which lowers the accounting costs.

For instance, even if it was not his own, Fibonacci introduced the multiplication, — here we present the *per gelosia* which made this operation more accessible to the most part of peoples who were able to read[49]. This methods appears in Europe only fifty years after Fibonacci's passing in an anonymous latin treatise written in England, the *Tractatus de Minutis Philosophicis et Vulgaribus*, but has been certainly developed by an arabian mathematician. For instance, if one want to multiply say 2345 by 467, as was exposed in the book, on can operate by associativity in the same way as with roman numeral — *i.e.* : one can note that 2345 = 2000 + 300 + 40 + 5 and 467 = 400 + 60 + 7.

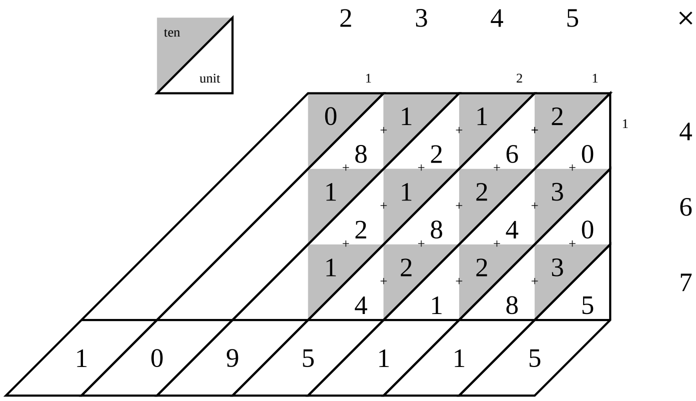
name that Pise decide to grant to him a salary of 20 lira. In fact this is a very complex subject since it seems that, Perizolo, notary of the Roman Empire, mentions Leonardo as Fibonacci in 1506 — for more details on that subject, see Drozdyuk & Drozdyuk (2010).

[48]That *Liber Abaci* should have been the only gate through which practical algebra has been transmitted to Italy and then christian Europe has been convincingly recently disclaimed by Høyrup (2011). One can mention at least two early tentatives to work with hindu-arabic numeral : the first one was Rabi Abraham ben Meir Ibn Ezra who, in his 1148 *Sefer ha-Echad* — Book on Unity —, borrowed the Indian place-value system, but instead of using the traditional number signs, represented each with the first nine letters of the Hebrew alphabet — keeping the Indian sign for zero — and Thomas Le Brun yet known also under the name Qaid Brun, an englishman who was in turn secretary of William I of Sicily "The bad", and later a reformer of the exchequer under Henry II, who seems to have tried to introduced them in the usage of the exchequer but has not been followed near by 1158 — here the conditional comes from the fact that we have find some recent sources on the subject attesting the existence of some documents which are not referenced. According to Smith & Karpinski (1911), in the same period one can find the first computations with hindu-arabic numerals in a german *Algorismus* of 1143. Fibanacci himself, in his introduction to *Liber Abaci*, tells us that, in what concerns hindu-arabic numbers, while in business, he went in all the places where they were studied and taught — *i.e.* : Egypt, Syria, Greece, Sicily, and Provence — and, according to the translation of Grimm (2002), he *pursued [his] study in depth and learned the give-and take of disputation*. There would have been also a montpelieran and a provençal connection.

[49]Some authors wrongly attribute this method to Luca Paccioli — see for instance Gleeson-White (2011). But, the earlier record attributes it to Ibn al-Banna' al-Marrakushi at the end ot tyhe 13th century. Nevertheless, it certainly has been independently developed in several part of the then known world, and there is any track of its employment by Fibonacci.

×	2000	300	40	5		
400	800000	120000	16000	2000	+	
60	120000	18000	2400	300	+	
7	14000	2100	280	35	+	
	934000+	140100 +	18680 +	2335	=	1095115

but we note that that method imply huge numbers multiplications. The *gelosia method*[50] of 9 by 9. This time too, one begin by a double-entry table but in each cases one find the unit, the ten, the hundred, the thousand. . . then one operate as indicated in the following example.



In what concern the division, he introduced to the european public the *galley method* which is obviously of chinese origin — see Lay-Yong (1966)[51].

Suppose one wants to divide 7385 by 214. The method, which is here decomposed stage by stage, began as shown here under :

①

7385 |
214

As $2 \cdot 3 < 7 < 2 \cdot 4$, one write 3 in the right place, write 2 under the 7 and the rest in the difference between 7 and $2 \cdot 3$ above the seven and do the two same operations for the one crossing the numeral already used.

[50]According to the dictionary, the name comes from the *gelosia window*, a window compose of wooden louvers set in a frame. The louvers are locked together onto a track, so that they may be tilted open and shut in unison, to control airflow through the window. *Gelosias* are reputed to permit to people to see unnoticed.

[51]This method stays nearly for three centuries the lone method taught, since it is paper economizing and that the paper was first imported in occident in the 13th century.

②

```
1
7385   | 3
214
```

③

```
10
7385   | 3
214
```

Now $3 \cdot 4 = 12$ but on the line there is only 8 so we must borrow a 1 to 10. The remainder is 9 and the remainder of 18 -12 = 6 so

④

```
 9
102
7385   | 3
214
```

Now, as we have exhausted the use of 214, we rewrite it shifted by one position.

⑤

```
 9
102
7385   | 3
2144
 21
```

And one begins again.

⑤

```
 1
 9
106
7385   | 34
2144
 21
```

⑥

```
 1
 92
106
7385   | 34
2144
 21
```

⑦

```
 10
 98
1069
7385   | 34
2144
 21
```

At the end of the process, one find that there is 34 times 214 in 7385 with a reminder of 109. This method generate at least two remarks: first, it is not self-evident for a learner to understand that a number couldn't be written on the same line, but this is marginal. Secondly, there was no attempt to try to make any expansion of the remainder in decimal value as one uses to do nowadays. It was not in the spirit of the time and, even if it has been know in China and Persia long ago before Fibonacci, even if it was developed in Europe in the 14th century writings of Immanuel ben Jacob Bonfils, it has been necessary to wait the 17th century and the success of the *De Thiende* of the flemish mathematician Simon Stevin[52] so that the decimal development becomes an universal practice[53] — see Stevin (1935[1585]). But, from a practical point of view, that was perfectly justified. In a tradition inherited from the egyptian, it was the usage to write a non integer number as A.bcd in the form :

$$\frac{d}{10}\frac{c}{10}\frac{b}{10}A$$

[52] To help the diffusion of his thinking, Steven had translated his work by himself.

[53] But, one more time, we know now that in the middle of the 10th century, al-Uqlidisi of Damascus introduced place-value decimals to the right of the decimal point. Unfortunately, non one saw a particular reason to adopt it and this brillant idea sleeps for five hundred years before that arab scholars awake it and tree more centuries before Steven convinces europeans of its power — see Devlin (2010).

where the denominator of each fraction was understood as the product of all denominator value that preceded it[54]. For instance, Fibonacci would have written the result of the former division $7385 \div 214$ as $\frac{109}{214}34$. Suppose now that this value be a monetary value. In his living time, the subdivision of monetary unit was complex[55] but with the Parma system of account Fibonacci would have written $\frac{2}{20}\frac{6}{12}34$ that is 34 *Lira* 6/12 of *Lira* and 2/240 of *Lira* or in actual money 34 *Lira* 6 *Soldi* and 2 *Denari*.

An other innovation of the *Liber Abaci* is its advocacy for the use of negative numbers which Fibonacci interprets as debts or, in another work — *Flos*[56] —, throughout the deficit metaphor. Of course, even in that case, Fibonacci was just a diffuser since negative numbers were of chinese and, later, indian origin. And even his interpretation was not original since, in the 7th century, the indian mathematician Brahmagupta has yet called them debts :

> *A debt cut off from nothingness becomes a credit; a credit cut off from nothingness becomes a debt.*

The true innovations of the *Liber Abaci* could be find in its numerous applications and nearly all applications, one can find in that book, are of commercial and financial content. That is not to say that applications were not developed as a by product of the development of the Algebra and essentially from the work of Muhammad ibn Mūsá Al Khwārizmī from which Fibonacci transmitted essentially all the mathematical innovations but, the main difference is linked to the fact that, apart of his involvement in geometry, Khwārizmī's examples are all essentially linked to legal problems of inheritance. Without any means to know that by himself, there are evidences, through the identity of some developed examples, that Fibonacci commercial and financial investigations are linked to those of earlier indian mathematicians as Āryabhata who, overlapping on the 5th and the 6th century, developed some interest calculi in his astronomical book *Āryabhatiya* — see Clark (1930) —, Bhāskara whose commentaries on *Āryabhatiya*, written in the 7th century, include some problems related to partnership share divisions, and the relative pricing of commodities — Goetzmann (2004) — and Sridharacaryas who, in the 10th century, devote some of its 300 verse couplets *Trisastika*, to a few practical interest rate problems and a division of partnership problem — see Ramanujacharia & Kaye (1913). In *Liber Abaci*, one can even find a 9th century copy & past problem borrowed to the jain mathematician Mahavira's book, *Ganita Sara Sangraha*, where three merchants find a purse lying in the road. The first asserts that the discovery would make him twice as wealthy as the other two combined. The second claims his wealth would triple if he kept the purse, and the third claims his wealth would increase five fold[57].

Try to imagine how to determine an unknown price from a given quantity of merchandise when the price per unit is known[58] — *suppose a 100 rolls costs 40 lira, how much would five rolls cost ?* — without the apparatus links to the resolution of a first order equation with one unknown.

Nowadays, one would write :

$$x \cdot 100 = 40 \implies x = \frac{40}{100} \implies x \cdot 5 = \frac{40}{100} \cdot 5 = 2$$

[54]It is clear that this way to write a number is of arabic origin since, under this structure, it is clearly written from left to right. One must also note that the fraction bar is a Fibonacci innovation.

[55]Until six subdivision as *Ducado, Lira, Grosso, Soldo, Piciolo* and *Denaro*, but there was no uniformity between towns. For instance, in Parma, the main money was the *Lira*, which was subdivided in 20 *Soldi* themselves subdivided in 12 *Denari*. It is notable that unit of weight and measure were even more complex.

[56]See Fibonacci (2010).

[57]In fact, nearly all, what in a near past, appeared to be Fibonacci innovations have been demonstrated borrowed to indian or arabic traditions. For instance, the celebrated Fibonacci sequence of the growth of a rabbit population has been borrowed to the *Chandahshastra — The Art of Prosody* — of the sanskrit grammarian Pingala, who was active somewhere between 450 and 200 BCE. Then, the indian mathematician Virahanka showed how the sequences arises in the analysis of metres with long and short syllables. Finally, around 1150, the jain philosopher Hemachandra composed a text on the term value of this numbers — see Devlin (2010).

[58]It's the subject of the chapter 8.

a solution that nobody can find remarkable since first order equations are taught in the beginning of the secondary school. Leonardo present it under a different structure as the famous *rule of three* — *i.e.* :

$$\begin{matrix} 40 & 100 \\ ? & 5 \end{matrix} \qquad \frac{4 \cdot 5}{20} (=?)$$

As remarked by Goetzmann (2004), the rule of three is one of the oldest algebraic tools since it appears for the first time in the *Āryabhatīya*, and is extended and elaborated upon in *Bhaskara*s commentaries, in which he applies it to problems quite similar to those analyzed by Leonardo — see Sarma (2002). It's so simple that we can hardly imagine how were perform exchanges before it came of common knowledge by every traders. Of all evidence, in every transaction, one of the two party was certainly defrauded. As a proof for that case, one can advance that, if it has not been the case, Fibonacci should not have developed so many complex examples on that subject. Immediately, Fibonacci explains how to find the price ratio between two goods

> *It is proposed that 7 rolls of pepper are worth 4 bezants and 9 pounds of saffron are worth 11 bezants, and it is sought how much saffron will be had for 23 rolls of pepper*

One more time for the common run of people living in the 21th millennium, there shouldn't be any difficulty with this operation : One find the unit price of the roll of pepper, the unit price for the pound of safran, then one calculate the price ratio and, after multiplication by the number of desired rolls, the mass is said, that is :

$$\left.\begin{matrix} 7 \cdot p_p = 4 \\ 9 \cdot p_s = 11 \end{matrix}\right\} \implies \frac{p_p}{p_s} = \frac{4 \cdot 9}{11 \cdot 7}$$

that is to say that if one wants to exchange 23 rolls of pepper against saffron one will obtain :

$$\left(\frac{4 \cdot 9}{11 \cdot 7}\right) \cdot 23 = \frac{2}{11}\frac{8}{7}10 \approx 10.75324675\ldots$$

But, on more time, it's only the application of the *rule of 5*, that Bhāskara and later Indian mathematicians developed for expressing price/quantity relationships across several goods — *i.e.* :

$$Saffron Bezants Pepper ?4791123$$

In a implicit way, Fibonacci teaches how, in that matrix, the balance is between the two diagonals

$$Saffron Bezants Pepper ?\textcircled{4}7\textcircled{9}11\textcircled{23} \quad and \quad Saffron Bezants Pepper ?4\textcircled{7}\textcircled{9}\textcircled{11}23$$

and finally give the thing — the ? in the matrix.

Up to this point, Fibonacci also shows how to arbitrate between moneys because, in that time, even in Tuscany, even if cities have inherited of the late roman money division — *denari*, *soldi*, *lire*[59] —, the relative value and metallic composition of the various moneys varied considerably through time and across space[60]. On that subject, after 1252, the teachings of the *Liber Abaci*, will became unavoidable, due to the introduction of the golden florin which was the first money which could simultaneously serve as a unit of account and transaction on a 1:1 basis, since there were evidence, that it was never debased — see Velde (2000).

But, as bring out by Goetzmann (2004), and echoed by Rubinstein (2006), it's in the finance field that *Liber Abaci* shows all the genius of its writer through the discussion of four type of problems :

[59] Which finally remains in the english pounds, shilling and pence system.

[60] It's the reason why, he also give many examples of minting and alloying of money.

① How to split fairly the profit of a joint venture when contributions are unequal, are made at different points in time, and in different currencies or goods and in cases in which business partners borrow from each other?

② How to calculate the profits generated by a sequence of business trips in which profit and expense or withdrawal of capital occurs at each stop?

③ How to calculate the future values of investments made with banking houses?

④ The first use of present value analysis as a criterium to evaluate investments, including specifically the difference between annual and quarterly compound interest.

In what concerns the first point, on must know that in the 13th century italia, the business ventures were organized through the *commenda* contract[61] which stated how the *commender* — the partner who invested his funds — and the *traveler* — the partner who invested his labor — should divide the profits. If this last one, doesn't abound to the financement, the *commenda* was unilateral and the *commender* retained 3/4 of the benefits. If the traveller decide to abound to the financement, in which case the *commenda* contract was reputed bilateral, it was generally up to half of the *commender* abounding.

Then, according to the specific dispositions of the contract, the study of notarial archives from cities as Barcelona, Genoa, Venice, Amalfi, Marseilles, and Pisa[62] reveals that, generally, any profit was usually divided 1/2-1/2 while the *commendator* bore 2/3 of any loss and the *tractator* 1/3. But variations of this dispositions could be encountered. For instance, in Dubrovnik, the share 3/4-1/4 of the unilateral *commenda* was not a rule at all.

Here, Fibonacci innovates in explaining how to fairly — that is according to the initial contribution — divide the profit when there is more than one *commender* — in those time, they constitute a *societa*[63].

In what concern the travelling merchant, here is how Fibonacci states his basic problem:

> *A certain man proceeded to Lucca on business to make a profit doubled his money, and he spent there 12 denari. He then left and went through Florence; he there doubled his money, and spent 12 denari. Then he returned to Pisa, doubled his money and it is proposed that he had nothing left. It is sought how much he had at the beginning.*

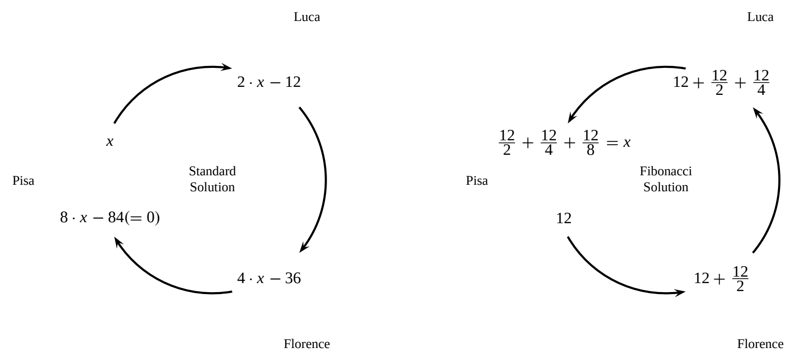
Here, one can find the solution original under the restraint that words have the same meaning for us than for Fibonacci, that is to say that one must understand that, in returning in Pisa the man spent also 12 denari and that the money double before the expense.

Here is how it may be reasonable to solve the problem by a forward argument. If X is the initial capital, one has for an initial capital of x, a capital of $2\cdot(x-12)$ in Lucca, of $2\cdot((2\cdot x-24)-12) = 4\cdot x-36$ in Florence and of $2\cdot((4\cdot x-36)-12) = 8\cdot x-84$ at his return in Pisa, which is finally equal to 0, process which give birth to the forward dynamic scheme

[61] Prior to Pryor (1977), it was common to place the origin of this contract in the muslim qīrad contract. Since then, the Jewish *'isqa* contract and the roman *societas* appear to be two other sources of the *commenda* contract.

[62] According to Fibonacci & Sigler (2003[1202]), the *Constitutum Usus* of 1156 is the the earliest surviving municipal document specifying the conditions of the *commenda* contract.

[63] For the specific operations, one can consults one more time Goetzmann (2004) or Fibonacci & Sigler (2003[1202]).



As finally, he has spent all his wealth, one must solve the first order equation $8 \cdot x - 84 = 0$ which give $x = 10.5$. This is how Fibonacci could have simply solve his problem.

But as the sign of a great spirit, he find the solution in following a complete new path, which could certainly be identified as the first backward argument in the history of thought: he actualizes.

He states that, since its capital double at each stop, its initial capital is 1/2 the capital he owns in Lucca, which is 1/4 the capital he owns in Florence, which is 1/8 the capital he owns at is return in Pisa. In terms of the initial capital. That is to say that a denaro spent in his third stay could not be evaluated on the same basis as a denaro spent in his second and in his first stay. So, 12 denari spent in his final stay is worth only 1/8 of 12 denari owned at the departure of the trip, 12 denari spent in is penultimate stay ar worth 1/4 of the denari owned at the departure and, 12 denari spent in his first stay are worth 1/2 of the initial denari. That is to say that, one could evaluate a flux of denari at different dates at the departure time. This give :

$$TotalExpense = \frac{12}{2} + \frac{12}{4} + \frac{12}{8} = 10.5 = InitialCapital$$

One must also remark that, at least by implication, Fibonacci's solution is linked to an anticipatory conscience of double entry bookkeeping since his solution discriminates clearly between capital and expenses in the sequences of accounts

Pisa

Ressources	Expenses
x	

Lucca

Ressources	Expenses
$2 \cdot x$	12
$2 \cdot x - 12$	

Florence

Ressources	Expenses
$2 \cdot (2 \cdot x - 12)$	12
$4 \cdot x - 36$	

Pisa

Ressources	Expenses
$2 \cdot (4 \cdot x - 36)$	12
$8 \cdot x - 84$	

One must note that, as simple that those accounts may seem, it was a formidable achievement to write it, since the successive resources incorporate negative numbers which he has just introduced in an earlier chapter. Now, in an exercise of experimental archeology, one can try to reconstruct his argumentation. As one can interpret negative resource values as vanished earning opportunities, they must slither from the Asset column to be written in the Liabilities column as shown hereunder.

Pisa → Lucca → Florence → Pisa

Pisa

Assets	Liabilities
x	

Lucca

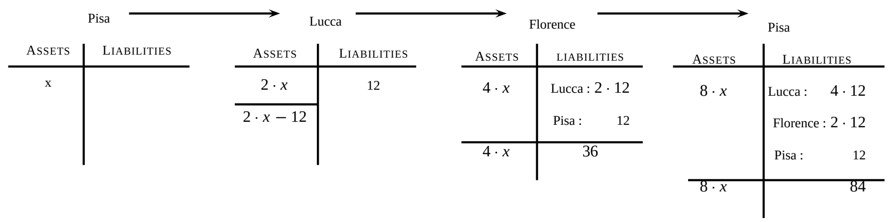
Assets	Liabilities
$2 \cdot x$	12
$2 \cdot x - 12$	

Florence

Assets	Liabilities
$4 \cdot x$	Lucca : $2 \cdot 12$
	Pisa : 12
$4 \cdot x$	36

Pisa

Assets	Liabilities
$8 \cdot x$	Lucca : $4 \cdot 12$
	Florence : $2 \cdot 12$
	Pisa : 12
$8 \cdot x$	84



Under this presentation, it is self-evident that the last Pisa's book is written in terms of accumulated assets, not in terms of initial assets. So Fibonacci must simply have divided all terms by 8 to, finally, obtain the return to Pisa book in terms of initial assets, that is :

Pisa

Assets	Liabilities
x	Lucca : $\frac{1}{2} \cdot 12$
	Florence : $\frac{1}{4} \cdot 12$
	Pisa : $\frac{1}{8} \cdot 12$
x	10.5

What a *tour de force* ! For the first time, since human beings tried to maintain the balance sequences of their commercial operations realized in distinct places, Fibonacci, from scratch, shows how to relativize all the entries and express them in initial value entitling the comparison of various complex flux of income. And more than that, to realize this *tour de force*, his mind, at least tacitely, should have been aware of the possibility to keep all the entries by the distinction between income and expenses — *i.e.*: A double-entry bookkeeping operations which, as one will explain later in this paper, will be truly available only two centuries later.

There is no evidence to state if Fibonacci has derived is invention of actualization, but it is more than a conjecture as can be seen from the problems which follow the travelling merchant problem. They are a set of sophisticated banking problems such as :

> *A man placed 100 pounds at a certain [banking] house for 4 denari per pound per month interest and he took back each year a payment of 30 pounds. One must compute in each year the 30 pounds reduction of capital and the profit on the said 30 pounds. It is sought how many years, months, days and hours he will hold money in the house.*

It is worth the effort to follow, helped by the modern apparatus, the effort of Fibonacci to find the solution of his problem. According to the traveller's one, one must reason as follow. First of all, one must find the interest rate and, according to the decomposition *denari*, *soldi*, *lire*, as a lire — a pound — is worth 240 denari, and the return of the investment is 400 denari per month for 12 month, that is to say 4800 denari or 20 lira/year, the rate of return is 0.2. Now, one has :

0	$100 - \frac{30}{1.2} + \frac{30}{1.2^2} + \frac{30}{1.2^3} + \frac{30}{1.2^4} + \frac{30}{1.2^5} + \frac{30}{1.2^6} = 0.234697$
1	$1.2 \cdot 100 - 30 = 90$
2	$1.2 \cdot 90 - 30 = 78$
3	$1.2 \cdot 78 - 30 = 63.6$
4	$1.2 \cdot 63.6 - 30 = 46.32$
5	$1.2 \cdot 46.32 - 30 = 25.584$
6	$1.2 \cdot 25.584 - 30 = 0.7008$

So there is a balance. And, as this balance is lower than 30, one knows that the account cannot stay open one more year. So one must reason in days because month are not regular unit. But to reason in days, one must also suppose that, all things equal, all the magnitudes are time homogeneous, that is to say that the man's daily expanses are equal to the $1/365$th of his yearly expenses and that the daily interest rate is also simply $1/365$th of the yearly one.

This carries to the new table :

0	$0.708 - \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^2} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^3} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^4} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^5} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^6} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^7} + \frac{\left(\frac{30}{365}\right)}{\left(1+\frac{2}{365}\right)^8} = 0.641613$
1	$\left(1+\frac{2}{365}\right) \cdot 0.708 - \frac{30}{365} = 0.629688$
2	$\left(1+\frac{2}{365}\right) \cdot 0.629688 - \frac{30}{365} = 0.550946$
3	$\left(1+\frac{2}{365}\right) \cdot 0.550946 - \frac{30}{365} = 0.471773$
4	$\left(1+\frac{2}{365}\right) \cdot 0.471773 - \frac{30}{365} = 0.392167$
5	$\left(1+\frac{2}{365}\right) \cdot 0.392167 - \frac{30}{365} = 0.312124$
6	$\left(1+\frac{2}{365}\right) \cdot 0.312124 - \frac{30}{365} = 0.231642$
7	$\left(1+\frac{2}{365}\right) \cdot 0.231642 - \frac{30}{365} = 0.15072$
8	$\left(1+\frac{2}{365}\right) \cdot 0.15072 - \frac{30}{365} = 0.0693538$

So, one has found that after 6 years and 8 hours the account will be nearly void — Fibonacci goes further, since he displays the exact answer which is 6 years, 8 days and $\frac{1}{2}\frac{3}{9}5$ hours in his own notation but one renounces to go to this stage not to tire the reader.

But what a fantastic operation! It outperform any performed computation before centuries. And from this problem, Fibonacci constructed 11 other examples from which the following has been extracted because it's has been viewed as the founding problem of the modern finance since, for the first time in history, not only the present value criterium is applied to discriminate between two apparently identical payment which differs by their sequence. It's the celebrated *On a soldier receiving three hundred bezants for his fief*.

The Fibonacci's story invented to expose this *sui generis* problem is the story of a soldier of whom the King want to reward his service record in granting him an annuity of 300 bezants/year, paid in quarterly installments of 75 bezants. Then the King alters the payment schedule to an annual year-end payment of 300 bezants. Knowing that the soldier is able to earn 2 bezants for one hundred invested bezants, Fibonacci asked if the situation of the soldier is better in the first or in the second schedule.

It's clearly a simple present value problem which can be analyzed in the first year of its payment. In the case where the grant is served in one shot, as the mensual rate of return that is accessible to the soldier est of 2%, the present value is :

$$V_{300} = \frac{300}{1.02^{12}} = 236.548$$

and in the case of the quarterly payment of 75 bezants, one find :

$$V_{75} = 75 + \frac{75}{1.02^4} + \frac{75}{1.02^8} + \frac{75}{1.02^{12}} = 267.437$$

So after alteration of the schedule, it comes that, in present value, the pension has been altered 30.8892 of bezants: The soldier should have been better off, if the king shouldn't have altered the payment schedule. As a by product of this accomplishment, and only as a by product, Fibonacci developed the rabbit population growth problem and as a natural consequence, the first analysis of a geometric sequence of number.

It could be tempting to minimize the stupendous financial accomplishment for Fibonacci pretending there is no news under the sun, since capitalisation is an operation which was used before the Hammurabi code that is 18th b.c.. It's a very simple operation which amount to compound the due interests at the expiration date. And, as shown by Goetzmann (2004), one can also be astonished by the incredible connection between the *Liber Abaci* and the babylonian problem exposed in the tablet 8528 conserved in the Berlin museum and published by Neugebauer (1935).

TABLET 8528

If I lent one mina of silver at the rate of 12 shekels (a shekel is equal to 1/60 of a mina) per year, and I received in repayment, one talent (60 minas) and 4 minas. How long did the money accumulate?

LIBER ABACI

A certain man gave one denaro at interest so that in five years he must receive double the denari, and in another five he must have double two of the denari and thus forever from 5 to 5 years the capital and interest are doubled. It is sought how many denari from this one denaro he must have in 100 years.

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If capitalization is necessarily a by-product of the loan operation, as demonstrated by the nearly three thousand rolled years between the first operation of capitalisation and the actualization by Fibonacci, that last operation is linked to far more complex mental schemes.

But, as necessity has a value of law, this great innovation couldn't stay a dead letter. And one knows that it has not been the case. Certainly, Fibonacci was a first class teacher who never spared himself in diffusing the new way to calculate without a mechanical support[64] or to promote his new financial instruments.

One can support this assertion by three remarks: First, during his life time, precisely in 1228, there has been a second edition of the book, which was an exceptional event for such a book, secondly his fame arrived early to the ears of the Fredrick II[65], emperor of the Holy Roman Empire, whose court

[64] It would be tempting to write here with a pencil and a sheet of paper, but this wouldn't represent Fibonacci's actuality.

[65] It seems that he has read by himself Fibonacci's book.

mathematician — John of Palermo — was charged to ask three problems to him[66], two of them been either later expended in his *Liber Quadratorum* — see Fibonacci & Sigler (1987) — or incorporated in his *Flos* — currently untranslated, at least from the latin transcription of Fibonacci & Boncompagni (1857-1862) — both published in 1225. Third, in his old age, in 1241, Fibonacci receive a pension of Pisa which acknowledged equally his effort in the two distinct field of the eduction of the citizens and dedicated service[67]. That is to say that, during his time life, Fibonacci helped his town in performing fructuous financial operations.

But the effort of a *lone poor mathematician* to transmit a new way to cope with number would have been a waste, if it shouldn't have been relayed by some other means because, first the editing technology of that time was so expensive that only the happy few could have access to books.

A book — initially named a *Codex* — was a technological progress on the *Volumen* — a scroll of papyrus — because it was far more durable than the later. It was design on the same structure that the assembly of wooden and wax tablets used for drafts and provisional texts. But they were also far more costly since the pages were made of parchment, that is to say sheep hide, even if parchment were apt to support reverse writings, support to be fold and sewed in registers.

But, more than that, books were hand written and, since this was a very costly time consuming operation, extraordinary costly. For example, Dittmar (2011) yield that in 1383, in England, a scribe was commissioned to write a single service book for the bishop of Westminster. For this work, the scribe was was paid £4, a sum equivalent to 208 days' wages for a skilled craftsman knowing that the costs of illustration, binding, and paper were listed separately and that he performed the transcription part time and completed the project over the course of two years. During the manuscript era, books were sufficiently rare and valuable that they were used as collateral on substantial cash loans extended to scholars by Oxford and Cambridge universities — Bell (1937).

Initially produced in the *scriptoria* of the friary, the most part of the codex was devoted to religious books. But, in the 12th-13th century, slowly, things began to change. In the secular world, the extraction of the scholar education from the cathedral and monastic schools to the Universities, from Bologna (1088), to Siena (1240), while passing through La Sorbone (1150), Oxford (1167), Palencia (1208), Cambridge (1209), Salamanca (1218), Padua (1218), Montpellier (1220), Naples (1224) and Toulouse (1229), created and expanded markets for law, mathematical, medical... books, began to develop due to a small literacy increase, causing the opening of scriptoria in towns. But this movement stayed marginal — less that 25 % of the manuscripts of this time are devoted to non religious subjects. So, a book like *Liber Abaci* could have be written only for a very small audience which was yet ready to receive his message because, to accept to be involved in such a purchase, one must have been persuaded that it was worth the expense. So the book was written necessarily for rich merchants and, more precisely, for those who were called to rule the city.

In what concern the replacement of the roman numerals by the hindu-arabic numerals, the audience was certainly not too difficult to convince, because with the development of exchanges with the arabic world, it went to be secure that the later outshine the former. After all, as recorded by Fibonacci himself, his father, who has been send in Bugia[68] to supervise the pisan commerce with this tunisian town, was

[66]The first problem is a problem looking like those resolved in *Liber Abaci*. The second was to find a rational number r such that both $r^2 - 5$and $r^2 + 5$ are rational squares for which, Fibonacci established that if $r^2 - n$ and $r^2 + n$ are rational squares, then there is a right triangle with rational sides, hypotenuse $2 \cdot r$, and area n, which conduct to the solution $n = 5$. In what concern the third problem — find a root of the cubic equation $x^3 + 2 \cdot x^2 + 10 \cdot x = 20$ —, according to Brown & Brunson (2008), it was borrowed to Omar Khayyams *Al-jabr* — see Kasir (1931-1972). Fibonacci proved that the solution of this cubic equation was neither an integer, nor a rational, nor a number of any of the forms from *Book X* of Euclid's *Elements*. So he decide to approximate it. Unfortunately, he failed to give a good approximation since the number he gives was neither a truncation nor a rounding up of the actual root — for more on that subject see Brown & Brunson (2008) who try to explained why he made this mistake.

[67]See H. Lhüneburg post at `http://www.mathematik.unikl.de/~luene/miszellen/Fibonacci.html`

[68]Now Bejaia in Algeria. In those times, Bugia was one of the main intellectual centers of the arabic world of equal fame with Sevilla and Toledo. Strikingly, the last crusade which ended under the walls of Tunis after the passing of Louis IX of France by a treatise between Charles d'Anjou and Tunis's emir which bestowed, among other things, to christian the right to free trade in the

persuaded that his son could, to some purpose, find some interest in learning the arabian way of computing. There is no reason to doubt that he was not alone of his kind, even if he was the lone to have a son which became able to convert it in a scientific revolution.

Secondly, tuscany merchants involved with mediterranean trade were certainly already ready to cope with the hindu-arabic numeral. Because, if from 1095 to 1291, the crusades failed to hit their main objective, which was, in the time terminology, to free Jerusalem of the faithless, despite the recurrent conflicts which will kept on going between christians and muslims, they will have, as unintended consequences, to secure the wealth of merchant italian cities. Their wealth began by the provision of ships to transport many of the crusaders to the Middle East and set up with the trade perfumes, spices as saffron, jewels, silk, dyes, tapestries, ivory and other products which the Europeans gradually came to value[69].

The croisades influenced also the intellectual development of Europe. Above all they liberalized the minds of the crusaders which were in contact with the leading science from which they had so much to learn. In particular, certain great spirits began to untie from the aristotelian tradition endorsed by the catholic church. One can easily conjecture that, in that manner, the ground was set for merchants to endorse hindu-arabic numerals.

And Fibonacci has been capable to convince his pisan fellow citizens, the inhabitants of the other main toscan and venetian cities and, finally, the world. But this will take more than a century and at the end Pisa would have been absorbed in the florencean *contado*[70]. The more astonishing is that, at least for a contemporaneous literate man, *Liber abaci* was a big and difficult book whose messages could only be assimilated by peoples ready to set aside the ancient way of doing calculation.

3 The diffusion of the new credo

It is clear that the simple exposition of the hindu-arabic numbers would had not suffice to guarantee their universal adoption. The singularities of Gerbert of Aurillac, of Adelard of Bath, and of Abraham ibn Ezra, all of whom failed to make school of their innovative approaches pleads in that direction.

In a certain way, the occidental mind was closed to innovation far before that the catholic dogma decreets that all truth must come from the sacred books — the bible and the four gospels. This in fact was a consequence of of a direct interpretation of the paulean anthem. Because, even if we know nearly nothing on the commitment of Paul with the greek philosophy — mainly Platon's one, since it supreme form was identified with the supreme good by Philo of Alexandria —, the weakness of his rhetoric in debate against greek philosophers, drives him to use a highly emotional argument in stating that non-christian philosophers *made non sense out of logic* [Romans (1:21-22)] and that *the wisdom of the world is foolishness to God* [Corinthians(1:25)]. This was unfortunate since as Paul's thought became the main authority on christianity, it induces a rejection of any rational thought to be substitute by authority, magic and mystery — see Freeman (2003).

As an exemple of the consequences of the adoption of Paul's anthem, and opposite to what was believed for a long time, Alexendria's library has not been destructed as a byproduct of the 642 muslim conquest under the caliphate of Omar ibn al-Khattâb, but by the joint action of the 48 B.C. Caesar conquest who, incidentally, set the fire to the main building, by the 270 A.D. Aurelian attack whose aims was to suppress the revolt off the queen Zenobia of Palmyra, which destructed the main library and, most of all, by the bishop Theophilus and his nephew Cyrille christian fanaticism which, in 391 A.D., supporting the proclamation of the Christianity as the formal and sole religion of the Roman Empire and the interdiction to practice any other cult by the emperor Theodosius in 391 A.D., conduct to the complete

Tunisia territory. But in that time, Fibonacci was dead since 40 years.

[69]On that subject and, more precisely, on the role and luck of Pisa in the mediterranean trade, one could consult Tangheroni (2002) and Tangheroni (2003).

[70]During the Middle-Ages, in the north of Italy, the *contado* was the territory under control and dependant of a major city. Pisa, short of its maritime power, will loose its independence in 1406.

destruction of the Serapeum — the former annex of the main library — see and the ignominious death they command against its head, the famous mathematician Hypatia[71] — Qassem (2008).

To change the main way to operationalize the algebraic calculations, a shift of paradigm was mandatory. Of all the obstacles which could get up to prevent the universal adoption of the new numbers system we have already raised, the local commercial customs were certainly not the least. But, with the development of the international trade during the 11th and 12th centuries, and the interaction with highly literate merchants as where oriental merchants who could maintain cheap account books given that they had the cheep egyptian produced papyrus to record transactions[72] there was certainly other traders than only Fibonacci's father who were aware of the improvement which would be bring by a standardization of their methods of calculation with those adopted by their Muslim correspondents with whom they traded[73]. For those traders, it was no more a question of fear to defy a prohibition, but a question of efficiency. For instance, roman numerals are in base 10. In the Fibonacci time, the lira was also in base 10, but this was not the case for it subunits, the soldii and the dinarii,which were in base 12. For those used to manipulate those units, there was no real problem as was the case until1971 with the english pound since it was corresponding only to a change of denomination — pound, shilling, pence — which date of the reintroduction of this roman system by Charles the Great.

Fibonacci could have certainly been genovese, venetian, anconian, ragusan rather than pisan since all those but he was pisan and he gave to his city his loyalty and his enthusiasm. And, for this, his city was grateful to him in voting, around his seventy years[74], a 20 lira pension *considering the honour and profit of* the *[our] town and its citizens, deriving from the doctrine and diligent services of the wise and learned master in abacus problems and estimates, useful to the town and its officials and in other things when necessary*, adding that they *ask him to serve the pisan municipality and its officials in the practice* of abacus.

We have no testimony of the way Fibonacci convinced his fellow countrymen to adopt the hindu-arabic numbers, and so everything is only guess. As already indicated, he operated in a favorable ground which was strewed with conservatism and illiteracy[75] and were books production was very costly. He could not expect to diffuse hindu-arabic numbers through his *Liber Abbaci*, whose price was exaggerated for the average trader.

Happily for us, in those times Frederic II of Hohenstaufen ruled the Holy roman Empire from Sicily — 1112–1150. According to a contemporary chronicler, he was the *stupor Mundi*, a king of exceptional culture and curiosity even if looked throughout the time lens. And when in 1227, he visited Pisa to sustain the city in its conflict with Genoa at sea and with Florence and Lucca at land, he asked to meet personally Fibonacci whom he knew at least for more than five years earlier how he was a great mathematician.

In fact, since his came back in Italy, Fibonacci was in epistolary relation with the main members

[71] According to Qassem (2008), the earliest sources of the arab version of the end of the library appears only appear only at the end of the 12th century A.D. that is nearly six centuries after the asserted event according to which the library should have been destructed. According to the account of Jamāl al-Dīn ibn al-Qiftī, an arab historian of the 12th century, when Amr ibn al-ʿĀs, the conqueror of Alexandria asked the Caliph Umar ibn al-Khattāb what should he do of the numerous rolls he founds in the library, he receive as an answer that *if what is written in them agrees with the Book of God, they are not required, but if it disagrees, they are not desired. Destroy them therefore*. So he decides to burn them which took six month. But, the historians of the generation of the conquest

[72] For more details see Lieber (1968).

[73] The merchant of what has been called the maritime republics where aware off all innovation. For example, in the 12th century, Pisa appropriated the unique subsisting copy of the Justinien Pandect, which she(it) had stolen from Amalfi — another maritime republic, which passed under the pisean control in 1137 — and at the end of the 13th century, Genes and Pisa were the first cities to develop the portulan charts that is some maps based on compass directions and estimated distances observed by the pilotes at see.

[74] There is an ambiguity about the date of the decree since according to different sources, it dates either of 1240, 1241 or 1242. According to Drozdyuk & Drozdyuk (2010), the confusion arises because of the pisan calendar whose discount of time was ahead of the Florence schedule of one period, a singularity which was stopped when Pisa was incorporated in Florence's contado in 1509. This reference change did not occur simply once but several times and they were not always taken into account in all documents as was noted by Boncompagni in 1852 — see Boncompagni (2011).

[75] On that subject, one can read Pirenne (1929).

of the intellectuals' circle with whom Frederic II had surrounded himself: the court astrologer Michael Scotus[76], the court philosopher Theodorus Physicus and the court astronomer Dominicus Hispanus. This last one was the dedicatee who in 1120 has suggested to Leonardo to write his second book *De Practica Geometrie* — Fibonacci & Hugues (2008) — dedicated to the analysis of problems of geometry in an Euclidian spirit[77].

Some years later, it was to the court mathematician and notary, Johannes of Palermo, to present to Leonardo a number of problems as challenges. The answers to three of these problems will give in 1125 *Flos* — see Fibonacci (1857-1862) — and, the same year the *Liber Quadratorum* — see Fibonacci & Sigler (1987)[78]. In *Flos*, he was able to find the the approximate positive solution of the equation : $x^3 + 2x^2 + 10x - 20 = 0$ in base 60 — *i.e.* :

$$1 + \frac{22}{60} + \frac{7}{60^2} + \frac{42}{60^3} + \frac{33}{60^4} + \frac{4}{60^5} + \frac{40}{60^6} \approx 1.36881$$

in a time where, at least in Europa, the best disponible approximation of π the one of Platon — $\sqrt{2}+\sqrt{3} \approx 3.14626$ which is exact only in the thousandth[79]. All this stuff was clearly constructed for the rich and cultivated. But, in his fifty years, Fibonacci should have had already constructed a strategy to diffuse the practical part of his knowledge to the pisan merchants.

To change the main way to operationalize the algebraic calculations, a shift of paradigm was mandatory. Of all the obstacles which could get up to prevent the universal adoption of the new numbers system, the local commercial customs were certainly the highest. But, with the development of the international trade during the 11th and 12th centuries, and the interaction with highly literate merchants as where oriental merchants who could maintain cheap account books given that they had the cheep egyptian produced papyrus to record transactions[80] there was certainly other traders than only Fibonacci's father who were aware of the improvement which would be bring by a standardization of their methods of calculation with those adopted by their Muslim correspondents with whom they traded[81]. For those traders, it was no more a question of fear to defy a prohibition, than a question of efficiency.

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[76]The dedicatee of the second edition of the *Liber Abbaci*.

[77]In this book, Fibonacci gives some new theorems with precise proofs, shows how to to calculate the height of tall objects using similar triangles, presents practical information for surveyors and shows abilities which make of him a one of his kind for his time living.

[78]In this last book, Fibonacci exposes his method. There is a final work of Fibonacci, which date the same times : the *Epistola suprascripti Leonardi ad Magistrum Theodorum phylosophum domini Imperatoris* which, until now, has only been published in latin by Fibonacci (2010), but has been analysed by Horadam (1991).

[79]It is not known how he obtains his result, but Brown & Brunson (2008) has convincingly reconstructed the way he could have done and they conclude that, in all likelihood he use linear interpolation even if his approximate root is neither a truncation neither a rounding of the numbers one can obtain by linear interpolation. Its just in the half way between to stage of expansion.

[80]For more details see Lieber (1968).

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things when necessary, adding that they *ask him to serve the pisan municipality and its officials in the practice* of abacus.

We have no testimony of the way Fibonacci convinced his fellow countrymen to adopt the hindu-arabic numbers, and so the following arguments, whichever be their persuasion power, are only guess. As already indicated, he operated in a favorable ground which was strewed with conservatism and illiteracy[83] and were books production was very costly. He could not expect to diffuse hindu-arabic numbers through his *Liber Abaci*, whose price was exaggerated for the average trader. A less expensive book was mandatory to help to diffuse what Fibonacci thought as the good practice for numbers, and mainly through followers' formation who, in their turn, could re-hang the good word.

If he should have act alone, there would have been no revolution because, in those times, education was restricted to a minimum which was humanities skewed, and whose access was reserved to a few number of peoples. At the primary level, even in Italy, one can say that no things that really matter have changed since, in 789, Charlemagne has enacted that every cathedral and monastery should open schools, reserved to a male assistance, where they could learn grammar, rhetoric, logic, latin, astronomy, philosophy and mathematics — essentially arithmetic and geometry. Essentially, young peoples learned to read and write in latin et vernacular languages — see Durant (1950).

In at last three occasions, he signaled that he has written a *Liber Abaci Minoris Guise*[84]. There as been a debate to know if it is really through *Liber Abaci* that hindu-arabic numbers have been transmitted to the western civilization — read for instance the dissent writing of Høyrup (2005), Høyrup (2007) and Høyrup (2011) and for the pro-voices Franci (2003) and Devlin (2011). But until recently this book seems lost until Raphaella Franci decides to study a undated manuscript which occupies pages 1 to 178 of the library codex 2404. Clearly, this book has been formerly studied by Arrighi (1986), who published it in Arrighi (1989). Fortunately, the dates used in some of the problems described place its writing around 1290[85], which gives to the manuscript the status of the oldest one in popular language[86].

As Gavin & Schärling (2012) put it, very most probably, it was the duplicated lecture notes which the pupils had to copy out. If this text is almost a copied stuck by chapters 8 - 11 of *Liber Abaci*, in the reserve near that pigeons are substituted to rabbits in the famous problem which is at the origin of Fibonacci's fame,with the omission of the material on hindu-arabic numeral, an over simplification of the presentation of the handled problems, the algorithms of operations and the methods for calculating with fraction as if all this was known to the reader. Setting aside some minor details, the most important innovation in this *book* is the inclusion of three chapter dedicated to the methods of interest and the amortization calculations, this last subject being completely absent of the *Liber Abaci*. The chapter 14, titled *Rules of 'saldare ragine'* deal with some question of mercantile practices as how to handle payments made on different dates.

As underline by Franci through Devlin, Leonardo was the only 12th century mathematician able to envision such a new subject of this kind. So, it is more than reasonable to attribute him the paternity of the work of which the book was a copy. In any event, we have credible proofs of the significance of the two pedagogical texts of Fibonacci which two centuries later constituted the essential sources of inspiration of the authors of arithmetic manuals.

Now as apparently, the *Liber Abaci in Minor Guisa* was written before the 1220 decade, we can easily

[83]On that subject, one can read Pirenne (1929).

[84]In the *Libber Abbaci* itself, but also in his *Liber Quadratorum* and in his *Flos — both published in 1225*. According to Devlin (2011), there is at least an other reference to this lost book. It is in the *Pratica d'Arismetricha*, published in the 15th century, in which the anonymous author speak of it source as the *Libro di Minor Guis o Libro di Merchaanti.*

[85]Even is the book begins by the declarative sentence *This is the book of abacus according o the opinionof master Leonardo of the house of sons of Bonaçie from Pisa*, this is insufficient to believe in he fact that the author uses the Leonardo writings as a basis for the construction of its book.

[86]One knows that a cistercian monk, Ugo Ugurgeri, has written a *Libro di Nuovi Conti* in Siena a thirty years before the anonymous one we are discussing. But no copy remains and in all likelihood, it was either a copy or at least a derivation of the *Libro in Minor Guisa*.

imagine that Fibonacci used it as course notes. Even if there is no track of the fact that he opened a school, it is obvious that to pass on his message, he needed followers. But the first followers were more urgent to apply that to teach. The figure of the father could also has been too much present

Nevertheless, by slow degrees, and for at least four centuries, certainly following the exemple of Fibonacci himself, the first system of commercial private and/or public schools was organized in Tuscany. According to Ulivi (2002), as soon as 1265, a man named Pietro, who should have been a direct pupil of the maestro, appears as witness in a bill of sale. This Pietro had a son and both of them were *maestro d'abaco*. They certainly taught in private school. During the last quarter of the 13th century, one can testify the existence of some communal abaco schools, the oldest testimony coming from the municipality of San Geminiano who hired a Michele in 1279. In 1282, Bologne institutes its own school. In the first rotuli of Bologna University, in 1384,1388 and 1407, one find that a Antonio Bonini Biliotti, a university professor, should lecture in arithmetics, and geometry at the pre-universitary level, a decision which will stay in vitality at least until the middle of the 16th century. In the 15th century, Scipione del Ferro, who was the first to use the imaginary numbers[87], would be one of the *Maestro del Abaco*. A similar situation appears to have been set in Perugia where in 1389 and 1396, it is attested that, near by the secondary school professor of grammar, there was a professor of geometry and abaco. Interestingly, as it can be deduced from Ulivi (2002), the municipality had deliberated a first time in 1277 on the opportunity to create a school but the decision was gained only after the third deliberation in 1285. During their life time, the Verona schools will see the teaching of Tartaglia and of Francesco Feliciano de Lazisio both in the 16th century. It's only in 1399 that Pisa hired two professor for the creation of a local school.

One must also notice that small commercial towns picked out a public school system contrary to the great ones, as Florence or Venice which opted for a private one. According to Black (2004), Florence never followed the practice of other tuscany towns which propounded academic subventions to their citizens. This certainly explain the differential treatment of the professors which was dependent of the fortune of the cities where they taught.

In Venezia, one disposes of testimonies that that give evidence of the presence of *Abaco Schools* since 1305 and, as time goes by, the number of students of those schools could have reached a little less of 150. The most famous one was the *Scuola di Rialto*, established in 1408, where instruction covered logic, natural philosophy, theology, astronomy and mathematic, and the teaching was of such good level that it was a kind of foundation courses for being a candidate to the University. And its least title of glory was to have had Luca Pacioli as pupil. Around 1360-62, Leonardo da Vinci received a formation in his birth town just before being send in Florence to enter in Andrea Verrochio's studio. If one doesn't know with precision the birthday date of Piero della Francesca, one knows that, before 1430, he was a pupil of a scuola del abaco.

According to the memories of his father Bernardo, in 1480-1481, Niccolò Machiavelli studied also in such a school — see Machiavelli (1954[1488]).

In what concern Florence, for which we have testimonies since 1283, the reputation of the dispense teaching was so high that the city trained the most part of the *maestri del Abaco* who teached in Toscany and Venetia for at least tree centuries. Between, the end of the 13th century and the first third of the 16th century, nearly sixty maestri have been counted which operated in twenty schools[88]. For the world education standards of those times, Toscany appears truly outstanding. According to the chronicler Giovani

[87]According to Cardan, del Ferro find the universal solution to second order equations. But et never published it since, in that time the university mathematical pulpit was regularly put in competition, according to a rituel where the resolution of arithmetic problems was mendatory. Dying, he bequests his solution to his son-in-law, Hannibal de Nave, and one of his students Antonio Fior. This last one in competition for a university vacancy with Nicolo Fontana — *Tartaglia* that is to say the stammerer — was unwise in asking this alter one to resolve too much third order equations. Tartaglia guessed the existence of this universal solution and finally found it. Then Cardan, called to care the agonising Tartaglia, convinced him to transmit to him his knowledge. Tartaglia accepted under the strict condition that Cardan kept the solution for himself. Fortunately for us, Cardan doesn't respect his parole, but being one of the best physician of his time he really cure Tartaglia who recover his health and the controversy began.

[88]Ulivi (2002) gives a very elaborate description of the implantation of the schools in the diverse districts of the town.

Villani — see Villani (1906[1341]) —, quoted in Davis (1965), that approximatively 10 % of Florence's population, that is to say between 8000 and 10000 boys and girls were being taught to read. One fourth of the boys — between 1000 and 1200 — decide to study in one of the six scuola del abaca[89]. The social basis of recruitment was broad from patriciate to small shopkeepers. And, as noted by Goldthwaite (1972), they learned their lessons well :

> *anyone who is familiar with the complexity of the arithmetic problems of their monetary systems, international exchange rates and other accounting and business practices and who has tested the accuracy of the calculations in their commercial recors can vouch the effectiveness of the training Florentines received in* scuola d'abbaco.

One must also note that the black plague which, in 1348, spread from Genoa to Rome invading Toscany, ravaging Florence to finally reach the papal states, only slowed down the development of the merchant educational system[90]. On the content, duration and organization of the studies in the scuoli, we are pretty well documented, but one throw back to Goldthwaite (1972), Ulivi (2002) and Black (2007). One must shortly underline that those schools were perceived as high school accessible with the prerequisite of the attendance to a primary cycle where children learned to read and to write. In average, the school days began at 10 or 11 years and last two days but according to the competence and abilities of the child, it could variegate accordingly.

After the school, they enter as novice in the trade but not only because swiftly others corporations find that the formation dispensed in the scuoli could represent a good prerequisite. Even if geometry had a very small part in the teaching of the scuoli, this part was sufficient and largely outperform what was taught on that subject in other places all over Europa. For instance, Zervas (1975) argues convincingly that the design of the florentine baptistery door of Santa Maria del Fiore by Andrea Pisano owe much to the abaco teachings.

Nevertheless for a life expectancy of less than 25 years to enter in the trade at the age of thirteen years after two years of formations was the most than can be demanded in those times.

To help them in their teachings, some of the maestri departed from Fibonacci and had draft new books.

4 Was Fibonacci's present value truly an idea of genius ?

We would be pleased if we were able to characterize Fibonacci's present value as an idea of genius. For such a task, we see two road which can be followed : to evaluate the impact on our civilization of the introduction of the present value or to verify if, even in a civilization where the degree of education is hardly comparable with the education even of the honest man in the time of Fibonacci, peoples can easily replicate the path he followed to invent the present value.

4.1 Is it so easy to replicate Fibonacci's reasoning ?

To cope with this question, we have post the web, a simple questionnaire and asked a huge lot of people to try to answer to two simple questions under a complete anonyma guaranty. But as many peoples have been contacted by direct contact with one or the other of the two authors of this paper, we can proudly say that there is at least a Field medal recipient in our panel. Undoubtedly, our data base is biased tower higher education peoples, but this was an intentional choice since we wanted to test if even educated peoples have some difficulties to follow the Fibonacci's path in solving his problem.

[89]The others, either came in one of the fourth latin school to study grammar and logic — nearly one eight —, in rethoric school, other in canon law school — certainly the cathedral school, and the last part, was taught in notarial schools. This crumbling of the formation was certainly linked to the fact that, in 1321, Florence was fighting to found a university.

[90]During the black plague, the most part of the great cities had lost more than 40 % of their inhabitants. Florence lost more of the third of its population in the first six month, and between 45 % and 75 % in the first year.

189 answers were collected for our treatment mainly from french peoples. Among those peoples who, kindly, participate, only one was aware of the *Liber Abaci*. The most part of the peoples who have heard of Fibonacci meet his name through Fibonacci's rabbits problem and the famous series of numbers it generates, one of them being aware of it's use in finance[91]. Only a small, but significatif, number of peoples — 17 — find the problem complex, the other finding it either easy or trivial. 33 peoples from many level of education including university professors in the economic field, fail to display the good result mainly because they do not devote enough time to the calculations. That is not to say that they do not follow a good reasoning and the main part of those who failed could have find the good result if they have more seriously considered the answer.

In a *prima facie* spontaneous answer, only 25 peoples follow the Fibonacci's path, the main part of the answers using the embedding of first order linear equations which is the natural way to cope with this problem since all the panel members were educated people and first order linear equations are taught in the secondary school whichever be their chosen specialization — scientific or literary. 4 peoples answered throughout a serie calculation. As peoples who have a training in mathematics, in economics and in management, with nearly a certainty, should have had a contact with backward induction during their training, it was not a big surprise to discover that only 4 peoples with either a humanities or with a paramedical or medical training followed a *prima facie* Fibonacci's path answer — they were only 4, after elimination of those with a law training, since those peoples could have had a former contact with the other field.

Later in the questionnaire, we asked if the people could imagine an other path to the answer, disclosing the fact that such a path could exist. Of course, this was a question for those who have answered to the former question through the embedding of first order linear equations. Only 14 peoples were able to find the Fibonacci's path. To two notable exceptions, all taught.

From all of this, we can conclude than the Fibonacci's path is not a trivial one because, even if the high educational level which characterized our panel create a bias toward the use of the embedding of first order linear equations, when the signal for another approach is given only a few peoples find the Fibonacci's path.

4.2 The impact of the present value on our civilization

An other way to evaluate an idea is toward it's impact on the civilization. But in the Fibonacci's case the impact of the present value is indissociable of the introduction of the hindu-arabic number because without their introduction our civilization couldn't have evolved as it has done.

First of all, contrary to the economists' anthem, developed by authors as Rosher (1878), before Fibonacci, interest rates could not effectively reflect the marginal productivity of capital since they were administered and in such could not be linked to the present value of its generated flux of futur incomes, because capitalization was essentially approximately calculable and present values, which are the at the core of the market evaluation of the interest rates were not in the conscience of those whose activity should have depend dramatically from it. So it look strange that authors as Heichelheim (1958) and Foster (1995) developed their understanding of ancient interest rate along an economic line of an equilibrium between demand and offer. That is not to say that in other case the economic line must be discarded, but in the case of the interest rates fixation before the seminal Fibonacci's work it could not be pertinent.

In fact, as convincingly advanced by Hudson (2000), ancient interest rates were administered on an avoiding calculation basis[92] — excepted for commercial loans. If it has not been the case, how could we

[91] Amazingly, the most famous accomplishment of Fibonacci was only an exercise incorporated in *Liber Abaci* to entertain the reader and to distract him from too repetitive exercices. It's only after it's study by Lucas (1977) at the end of the XIXth century, that this sequence became famous.

[92] According to Nemet-Nejat (1995), division were rarely perform in the sumerian scribal school. Scribes preferred to use multiplication with the help of inverse tables. In fact, whichever be the operation, scribes used tables to make their calculations easier.

explain that, in Mesopotamia, the interest rate remained stable to the rate of 1/60th by month from the law of the Third Dynasty of Ur — around 2000 BC — to Babylonian times — 625-539 BC? That in Rome, for many centuries it was fixed to 1/12 — *i.e.*: one *ounce* per *as* ? And that in Greece, bankers offer a rate of 1/10th on deposits — see also Homer & Sylla (2005) for other region/time long term stable interest rates?

All those rates seems to have been decided for ease of calculation, since the 1/60th rate corresponds to a time were the number basis was sexagesimal, the 1/10th to a decimal basis and the 1/12 to a duodecimal one. In such a way, the interest rates were not the product of an equilibration process but simply reflect the cosmogony of their time. The main clue in favor of this argument is that Sumerians were perfectly aware of the fact that herds do not grow at a rate of 33 1/3 percent per year[93]— according to Schmandt-Besserat (2002*b*) and Schmandt-Besserat (2002*a*) —, a primitive accounting system was working more that 7000 years BC. As the scribes were educated to write and to count, it was a natural consequence that in the time, they became able to approximate rate of return in various human activities. We must write approximate not only because of the number system they were able to use but, mainly, because cost accounting was yet to invent[94].

First of all, in ancient times, mainly debts were not stemming from advance of money for productive purpose, as convincingly advanced by Homer & Sylla (2005), but from non payment of arrears on various obligations as taxes or fees duly due principally to tax collectors or for consumption. As temples were collectors of debts, and as numbers filled a large part of the cosmogony of those archaic times, it was ineluctable that the value of the interest rates emerge from the birth metaphor[95] as it was applied to numbers — that is to say integers — and their fractional numbers. For instance, since the Old Babylonian period, the sexagesimal system has been chosen because 60 has the highest number of dividers amongst the number from 1 to 100 — *i.e.* : 60, 30, 20, 15, 12, 10, 6, 5, 4, 3, 2, 1 —, that is 12, which is a huge number in comparison to the 4 for 10, the 6 for 12… Sumerians used to symbolize the god heading their Pantheon — Anu — as 60/60 and, taking in account the inverse of its dividers, attributed 22 children to him — *i.e.*: 30, 20…2, 1, 1/2, 1/3,…1/30, 1/60.

In the similar spirit, Plutarch (1878) states that ancient egyptians associated the first integer pythagorean triplet [96] to the birth of their cosmogony. 3, the male entity, called Horus, was associated with the upright perpendicular ; 4, the female entity, was associated with Osiris; together they give birth to Horus, the hypothenuse 5, called by the pythagoreans the *marriage number*. This sexual anthropomorphism was later imported by romans. Belatedly, to their turn, christians create their own number cosmogony were, as God was associated with 1, when confronted with 0 they could do nothing but to devilished it[97].

Of course, it was easy to incorporate rational numbers. But, even if pythagoreans invent irrational numbers, they reject them because they were a failure to their own cosmogony. As we are fascinated by the greek mathematical achievements, we take for granted that their imprint pervaded to the lower class of their society — even in the case of the Alexandria's library, there are good reasons to think that its access was strictly regulated and that ancient as well as recent knowledge was jealously kept. But, this was far from being the case. More than that, as nowadays, intellectuals were subject to many gibes and often considered dead loads as attested by the *Philogelos* — see Hierocles & Philagrius (2001) — the

But what is of ultimate importance is that *every number remain an integer because the system was intended to serve only as an instrument of calculation*. That is to say, no calculation was really performed, up to the integers.

[93]Gelb (1967) has translated a cuneiform tablet on the growth of a herd of cattle which reveals this fact — see also Nissen, Damerow & Englund (1993).

[94]Cost accounting has been introduced in 1772 by Josiah Wedgwood — the first industrial pottery manufacturer and, incidentally, Darwin great-father — when, according to Gleeson-White (2011), he introduced the double-entry accounting to "undertake a rigourous and comprehensive examination of his firm's accounts and business practices".

[95]According to Hudson (2000), the word metaphor come from the concatenation of *pherein*, which mean to bear and *meta* which mean beyond in such a way that it means *pregnant with meaning*.

[96](3,4,5) is the first triplet which verify the pythagorean triangle equality $3^2 + 4^2 = 5^2$.

[97]All this being obfuscated by the incorporation of the jewish number's philosophy which, as incorporated in the Bible, was *de facto* incorporated in the catholic dogma.

fourth century greek jokes' collection which devotes more than the half of his jokes to *scholastikos — i.e.* : university students.

In a certain way, the occidental mind was closed to innovation far before that the catholic dogma decreets that all truth must come from the sacred books — the bible and the four gospels. This in fact was a consequence of of a direct interpretation of the paulean anthem. Because, even if we know nearly nothing on the commitment of Paul with the greek philosophy — mainly Platon's one, since it supreme form was identified with the supreme good by Philo of Alexandria —, the weakness of his rhetoric in debate against greek philosophers, drives him to use a highly emotional argument in stating that non-christian philosophers *made non sense out of logic* [Romans (1:21-22)] and that *the wisdom of the world is foolishness to God* [Corinthians(1:25)]. This was unfortunate since as Paul's thought became the main authority on christianism, it induces a rejection of any rational thought to be substitute by authority, magic and mystery — see Freeman (2003).

As an exemple of the consequences of the adoption of Paul's anthem, and opposite to what was believed for a long time, Alexendria's library has not been destructed as a byproduct of the 642 muslim conquest under the caliphate of Omar ibn al-Khattâb, but by the joint action of the 48 B.C. Caesar conquest who, incidentally, set the fire to the main building, by the 270 A.D. Aurelian attack whose aims was to suppress the revolt off the queen Zenobia of Palmyra, which destructed the main library and, most of all, by the bishop Theophilus and his nephew Cyrille christian fanaticism which, in 391 A.D., supporting the proclamation of the Christianity as the formal and sole religion of the Roman Empire and the interdiction to practice any other cult by the emperor Theodosius in 391 A.D., conduct to the complete destruction of the Serapeum — the former annex of the main library — see and the ignominious death they command against its head, the famous mathematician Hypatia[98] — Qassem (2008).

Obviously, the acknowledge main contribution of Fibonacci was to blow up the bolt which had been put on the safe which contained figures by disillusioning them. But, in reality, his contribution on the present value had an impact which was no less reformer of the civilizations. Until his seemingly harmless exercices, there was no way to take into account its flow. The present was dense and what would happen tomorrow had the same value as what occurred today. Or, for the first time in history, he shows that the time flow has in itself a value. In a way, he has de-densified the present since before him there was no other way that to add a futur and a present income.

Of course, it was only done for four periods of time, but we must remember that *Liber Abaci* was written toward teaching not practicing. Fibonacci could have easily extended the number of periods without difficulties since he was "the specialist" of Euclide in his time and therefore aware of the proposition 35 of the book IX of Euclide which states that *if as many numbers as we please are in continued proportion, and there is subtracted from the second and the last numbers equal to the first, then the excess of the second is to the first as the excess of the last is to the sum of all those before it*, which in modern mathematical language means that if a sequence of numbers $a = a_1, a_2, a_3, ..., a_n, a_{n+1}$ is in continued proportion — *i.e.* : $a_1/a_2 = a_2/a_3 = \cdots = a_n/a_{n+1} = r$ then $(a_2 - a_1)/a_1 = (a_{n+1} - a_1)/(a_1 + a_2 + \cdots + a_n)$ and conduct directly to the formula :

$$a_1 + a_2 + \cdots + a_n = a_1 \left(\frac{a_{n+1} - a_n}{a_2 - a_1} \right) = a \left(\frac{r^n - 1}{r - 1} \right) = a \left(\frac{1 - r^n}{1 - r} \right)$$

and if we note i the interest rate, we will find that the period one value of an income of a distributed from period one to n is :

[98] According to Qassem (2008), the earliest sources of the arab version of the end of the library appears only appear only at the end of the 12th century A.D. that is nearly six centuries after the asserted event according to which the library should have been destructed. According to the account of Jamāl al-Dīn ibn al-Qiftī, an arab historian of the 12th century, when Amr ibn al-Ās, the conqueror of Alexandria asked the Caliph Umar ibn al-Khattāb what should he do of the numerous rolls he founds in the library, he receive as an answer that *if what is written in them agrees with the Book of God, they are not required, but if it disagrees, they are not desired. Destroy them therefore*. So he decides to burn them which took six month. But, the historians of the generation of the conquest have not spoken of this fact. Its only six hundred years later, that this story began to diffused.

$$V_1 = a\,(1+i)\left[\left(\frac{1}{i}\right) - \left(\frac{1}{1+i}\right)\right]^n$$

in such a way that the current present value be :

$$V_0 = a\left[\left(\frac{1}{i}\right) - \left(\frac{1}{1+i}\right)\right]^n$$

If we set aside Archimede — see Heath (1953) —, who was the first to be able to evaluate at infinity the value a geometric series for a reason equal to two, this was not done before Oresme (1961) in the 14th century. So to apply Fibonacci's present value technic, it was necessary to make it by hand. This was certainly dedicated to the few ones who were able to operate the division who, nevertheless, stayed a difficult operation.

And with the success of the method, it becomes mandatory to dispose of tables to save time and efforts. Of course, as soon as 1340, Francesco Balduci Pegoletti — Balducci Pegolotti (1936) — published some accumulation of compounding interest tables[99]. A century later, one can find an arithmetical manuscript[100] written by Girolamo di Piero di Chardinale Ricellai, a member florentine merchant family, which contains tables computed by Antonio Mazinghi presented to exemplify some calculations on simple and compound interests.

But it will take a lot of time after the printing press diffused in Europa, at the end of the 15th century, to find an other table of interest. It will come through *L'Arithmétique* of the french mathematician Jean Tranchant in the middle of the sixteenth century[101].

The circumstances of the publication of the book are well known and as its the lone case of a financial exercise included in an algebraic book it deserves to give the circumstances. In 1547, Henri II of France inherits from his father François I of a colossal debt greater than the income of the tax system. He was thus obliged to break with the short-term loans traditional management of the french debt. Henri II, and his advisers, decides to launch a big loan to strengthen the existing longer-term debt and raise new funds. On March 15th, 1555, they launches a two million crowns — *écus* — loan reserved for the bankers-traders of the city of Lyon for an annual interest rate of 16%, with some strange rules which were certainly imposed by the hard tension linked to the situation[102].

Trachant willed to exemplify the problem encountered by merchants :

> *…En ce même ans avant la foire de la Toussaint il reçut aussi par les mains de certains banquiers la somme de 3954941 écus et plus, qu'ils appelaient le grand parti, à condition qu'il payerait raison de 5 pour 100 par foire, jusqu'à la 41-ième foire ; à ce paiement il demeurerait quite de tout ; savoir laquelle de ces conditions est meilleure pour les banquiers ? La première à 4 pour 100 par foire est évidente, c'est-à-dire on voit son profit évidemment. Mais la dernière est difficile : de sorte que les inventeurs de cette condition-là ne l'on trouvé qu'à tâtons et presque avec un labeur inestimable. Maintenant je veux montrer faire de telles calculations légérement et précisément avec raison démonstrative et facile à entendre.*

[99] One must note that the edition of 1936 uses only hindu-arabic numeral. But, Evans the editor certifies that the manuscript he uses for its traduction, which dates from 1274, uses roman and hindu-arabic numerals. The tables record the increase, at compound rate of interest 1, 1 1/2, 2, …, 8 % per 100 lires.

[100] Referenced as Palatino 573, in the *Bibliotheca Nationale* de Florence.

[101] On Trenchant we know nothing, which according to Sarton & L. G (1934) is rather strange for a book which enjoyed so much popularity.

[102] For example, The liberation of funds was effective in the fairs of the All Saints' Day 1554 and for the later to the Kings, Easter and of August feasts. The conversion of the pre-existent debt, with the aim to stretching out the duration, was compulsory. This obviously retroactive liberation of the funds was understandable by the consideration of the term of the pre-existent debts, for which the royal Treasury was not able to honor its signature. — see Gallais-Hamonno (2006) which is inspired by Doucet (1933*a*) and Doucet (1933*b*).

And to help merchant to verify if the new conditions made by the king made merchants worst off, he build some capitalization tables. Before Simon Stevin — Stevin (1958) —, in the 16th century, there will be no other published tables. Unfortunately, whichever be the cleverness of both authors, their tables are capitalization not actualization tables. Obviously, as we will see farther, until Stevin, actualization appears only as a good way to link the new arithmetic and algebra, not as a way to discount the future.

So before the 16th century, only elementary operations were involved in practical financial matters. If, in no way did Stevin revendicated the invention of the present value, it's certainly because he was aware of its existence. Even if, unfortunately, we know rather little of his early activities, it is commonly admitted that he was involved with financial activities[103]. It is through its work on interest tables, which was his first published work — 1582 — that he has realized the difficulty to do calculations with fractions and finally will gain the glory he deserves, introducing the usage of decimal number rather than fractions[104]. That is not to say that Trenchant himself was not aware of actualization since he quote a case where the rent of a farm is 500 per years — in arrears — for 3 years and it is desired to pay in advance by a single payment. Lewin & de Valois (2007) remind that he explains how to use the 4% table in reverse to find the sum which must be invested now in order to have 500 in a year's time. He then adds to this, the present values corresponding to payments of 500 in 2 and 3 years' time respectively.

In his preface to the *Principal — mathematical — Works of Simon Stevin*, Struik note that the publication of Stevin's interest table emulates the publication of other tables specially in the Netherlands. The first one to walk in the steps of Stevin was Marthen Wentzel van Aken in 1587 — see Wentzel van Aken (1594). He was a school teacher who responded to the solicitation of a merchant who found the book of Stevin to difficult. Then Ludolf van Ceulen[105] published the famous *Van den Circkel* — see van Ceulen (1596) — were he gives the best approximation of π since Al-Kashi[106], and some tables of compounding[107]. An amazing fact is linked to reason of the incorporation of interest tables in a treatise on the circle. According to Bosmans (1910), during a fencing lesson, a student raised a very difficult discount problem. Keeping a copy of the problem for himself, he had no respite before having solved it by himself, through the construction of the famous tables and, although being in very closed links with Stevin, he was persuaded to be the first to publish such tables.

Struik also quote a more difficult work to identify: the work of C. I. Broessoon — certainly, Cornelis Jan — whose tables, written before 1599, are those printed in the *Arithmetica, met een Tafel van Interest van een op 4 Hondert ende van 1/4 tot* $1/4$ *tot 12 op 't Hondetrt Interest op Interest per Jan Belot Dieppois* in 1629. Then one find the more elaborate tables of Ezechiel De Decker, the Gouda surveyor who was the great promoter of Stevin's decimal system and of Napier-Briggs logarithms — see Napier et al. (1626).

If the publication of *Liber Abaci* was contemporaneous of great innovations in the financing of the main italian city-states, the main of all being the creation of long term government debt[108], the work of

[103]According to Dijksterhuis (1970), he occupied a post in the financial administration of Bruges. According to Hadden (1994), the same Dijksterhuis has concluded from a casual remark in one of the works of Stevin, that he has been a bookkeeper and cashier in Antwerp.

[104]Honestly, Stevin's decimal numbers were not strictly his invention. Strictly speaking, it's a chinese invention. In Europe, was the jewish mathematician Immanuel ben Jacob Bonfils who was the first to plead for their utilization in the 14th century. The Stevin proposed very heavy notation such that 31.4567 would be noted 31⓪ 4 ① 5 ② 6 ③ 7 ④ were circled number must be understood as negative power of 10. This notation is obviously cumbersome for high decimal digits. It will be superseded by the lighter full stop in two time: First Franciscus Vieta began to use a vertical stroke to separate the integer from the fractional parts of a number. But this type of separation was already in use since the roman times. Then, it seems to be John Napier, in the beginning of the 17th century, who was the first to intentionally use the period as a decimal separator — some historian historians also move forward the name of Bartholomäus Pitiscus, a Napier contemporary — see Cajori (1928).

[105]On van Ceulen, according to Bosmans (1910), he was a perfect self-made mathematician, who own is living by teaching boxing and fencing.

[106]He made the number of decimals jump from 16 to 20 for the publication of *Van der Circkel*. Later, he adds 15 more decimals.

[107]It is notable that, unlike van Aken, van Ceulen does not quote Stevin as reference. It is perfectly possible that they were calculated before the publication of Stevin's book, but published more late.

[108]In 1171, Venice drew a forced loan from its citizenry called a *prestiti*. Paying a 5 % interest per year with an indefinite maturity, this bond, initially suspect to lender acquired rapidly the status of a valuable placement since Venice never default for

Stevin occurs also in the context of a great financial mutation in the public finance of the Netherland[109], the next publication of interest tables came from the bubbling early 17th century England[110]. In 1613, Richard Witt published his *Arithmeticall Questions*. Unfortunately, the book was quickly forgotten since he was the first work exclusively dedicated to the question of the interests. Richard Witt discussed nearly all what could be discussed about interest rates, exhibiting the formulas as a modern writer on the subject would do it to the notable exception of perpetuities.

This could seem particularly astonishing since perpetuities have been used early to by-pass the strict prohibition of the interest loan since Charlemagne.

As shown by Munro (2007), some carolingian monasteries invented the *census* contract remained until our days under the shape of the life annuity. Earlier, we have already encounter such a contract. To secure bequest of lands from the laity, monasteries guaranteed the donor surrendering all property rights the reception of an annual usufruct income — the *redditus* which give its name to the contract. This income could be in kind — for instance, as a share of the harvest — or in money. It was paid until the death of the legatee and some times could be pass to his heirs. As there were no final refunding, these contracts escaped the usury laws. Finally, this type of contract gave birth to two new contracts : first, one encounters the *bail à rente*, which corresponds to the sale of an immobile property for a perpetual annual income, and the *constitution de rente* also known under the name of *contrat à prix d'argent* in which the owner, keeping is property rights, sell, for a specific lump sum of money, the right to receive a fixed annual income[111]. Then in first quarter of the 13th century, during Fibonacci's life time, major towns began to sell such titles in two distinct declinations : the *rentes héritables* which were perpetual hereditary rents and the *rentes viagères* which went out in the death of the holder or exceptionally after the departure of one or two designed heirs often the spouse, the child or some close relative. Munro (2007) set to 1228 at Troyes, one of the major towns of Champagne fairs the first documented and successful urban sales of such rents and shows how quickly it diffused in Artois, Picardy and Flanders. In fact, even in a small scale, this type of public debt financing was practised in northern France and Flanders since the 1220s. France's kingdom will have the unhappy privilege to be the first state to use it to finance its debt, under the reign of François I.

We know today that the value of a perpetuity which deliver a C coupon is

$$V_0 = \frac{C}{i}$$

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more than a century. An first bond market was born. To pay their war expenses, Genes and Florence followed its steps. This is extendedly described by Poitras (2000). In that mater, Pisa was not outdone. The city began as soon as 1173 to made long-term debt commitments to creditors — see Favier (1971) — and according to Herlihy (1958), all along the 12th and 13th centuries, the finances of the city became more and more complex through capitalization of taxation and monopolization.

[109] All along the 16th century, Habsburg rulers of the Netherland wanted to increase his borrowing capacity to finance their many wars. To reach their goal, it was necessary to set up a fiscal reform insuring that the public debt could be adequately served. But rather than to go out weakened of these reform, provinces, and particularly Holland, were strengthened because, incredibly, they obtained a perfect control of the disbursement of the taxes. So Holland began to remove its former compulsory vouchers which had for effect to increase the trust in the decentralized provinces and allowed the appearance of financial markets. Finally, this successful fiscal politic conducted to the organization of the first stock exchange in the world in 1602 — for more details, see De Vries & van Der Woude (1997).

[110] The commercial needs of fresh capitals rising, in 1545, Henri VIII let the parliament to pass an "*Act Against usury*". For the first time, taking an interest of 10% year was permitted. But as the opinion was not prepared to such a great change, the act was repealed in 1552. In 1571, it was reenacted under the proviso of no reinforcement of the agreed interest from the lender, 10 % .staying the legal ceiling. In practice, the proviso was ineffective. The ceiling rate was steadily lowered passing by 8% in the *Act Against Usury* of 1624, 6 % in the *Act Against Excessive Usury* of 166, to fall to 5% in the *Act to Reduce Rate of Interest* in 1713. One must note that we must wait 1854 to see the repeal of the english usury laws. In 1571, this conduct to the foundation of the *Royal Exchange* under the patronage of Thomas Gresham which operated as a stock exchange but also as a place to exchange merchandises. London as a financial place was emerging in the beginning of the 17th century and the need for a better knowledge of the available financial instruments generates a big demand in practical and theoretical books dedicated to finance which, since, has never stopped — Lewin (1970) produces a list of 41 books on the subject of compound interest from 1618 to 1953.

[111] Those contract were widespread diffused specifically in Italy and in the aragonese kingdom.

a result which can be obtain in letting n going to infinity or by noticing that if we pay V_0 today it is to obtain $V_0 + C$ in one period of time. But if in the same period there is an other fixed income asset which deliver a interest of i%, in order that there is no arbitrage we must have[112] :

$$V_0 + C = V_0 + i V_0 \implies C = i V_0$$

And we know also that every fixed income may be analyzed as a combination of perpetuity. For instance, a n years constant depreciation property loan can be considered for the lender as receiving immediately a life annuity which will be put back in ten year which gives:

$$V_0 = \frac{C}{i} - \frac{C}{i(1+i)^n}$$

To our knowledge, it is not before the contribution of Edmund Halley, in the second half of the 16th century, that the formula explicitly appears — see Halley (1861)[113,114]. So, a perpetuity could not have been priced according to its formula earlier. But the least paradox is that, despite its widespread use, no one could have dare to pass to the limit, because since in the 7th century A.D., St John of Damascus has proclaimed God as being *infinite, eternal, without borders* and that *God, ..., is infinite and incomprehensible, and all that is comprehensible about Him is His infinity and incomprehensibility*, it was no more only Zeno's paradox which creates a tabu on the reflection on infinity. But, with Thomas Aquinas, in the middle of the 13th century and, especially, John Duns Scotus — the *subtle doctor* —, at the end of the same century, the reflection on infinity found slowly its pace in the even slower revival of the occidental thought, even if, contrary to Augustine who thought that God transcends the infinite — see Drozdek (1995) —, for Scotus, God is necessary, possess intelligence and will and is infinite. In fact the taboo of infinity was not really raised as far as only clergymen, because they were the only ones to have acquired the necessary mathematical culture, abounded in the debate.

We have to hold that the 14th century is the century of the *mathematical theology* in which *scientific* churchmen looked for their God through the mathematical concepts. This come from the fact that in the preceding century the aristotelian thought found a renewal of vitality. As states by Dolnikowski (1995), the basic tools of propositional logic and of euclidean mathematics, which accompanied this revival, help the thinkers to develop new original approaches to long-standing philosophical dilemmas. This was true for the main authors of the Oxford calculators of the Merton school, Thomas Bradwardine (1290 – 1340)[115], William Heytesbury (1313 – 1373)[116], Richard Swineshead[117] and John Dumbleton (∼1310 – ∼1349), true for Nicole Oresme[118] (1320-1322 – 1382). for Johannes de Muris[119] (∼1290 – ∼1351-1355).

In particular, it's because he was searching for the qualities of God, that Oresme, who, with an extreme depth, anticipated Georg Cantor's works of six centuries, attacked the convergence of the mathematical series. For instance, Oresme (1961), from Euclide, shows gives the good formula for the asymptotic value of the geometric progression for any value of a positive and less than one reason extending Archimede

[112] Even if there was no exposed formula, early in the middle-age, merchants were well aware of the possibility of arbitrage as explained in Poitras (2010).

[113] And, obviously, it's an incidental formula which is used to calculate finished life annuity, the objective of Halley being to demonstrate how logarithms can be used in this matter.

[114] In the beginning of the 19th century, it was a commonly known formula useful since the consolidation of the british debt in 1760 which created what is now known as *consols*.

[115] Archbishop of Canterbery for a month — death because of the plague.

[116] Chancelor of the University of Oxford.

[117] Fellow of Merton College (1340-1354).

[118] Evêque de Lisieu.

[119] Jean des Murs.

result which was obtain only for a reason of 1/4. He was also the first to show that the harmonic progression diverge[120,121]. In the same time, Bradwardine will have distinguished the geometrical growth — exponential — of the arithmetical growth by denying the Aristotelian solution of the problem of the movement—as its driving strength and the middle resistance the determine its speed[122].

This fascinating process of investigation will have finally as objective to make accept the notion of infinity and in spite of the lightning intuitions of Oresme who anticipated of numerous development of the analysis, it will be only with François Viète and Grégoire de Saint-Vincent whose works overlap the 16th and 17th century, Evangelista Torricelli, Pietro Mengoli and John Wallis in the 17th, that the path for a rigorous analysis of the series will be cleared enough so that a serious and dedicated analysis of the series can begin to be set up.

So, contrary to the speculation of Goetzmann (2005), there are no obvious links, between the asymptotic of the present value theory and the impetus which characterized the analysis of the series.

In return, where there is obviously a link between the solution of the cubic and quartic equation and the abbaco movement because for the first time since antiquity there were involved in the solution of problems whose origin was not geometric as for the Delian squaring of the cube problem but in financial analysis[123]. In spite of the problem posed by Tranchant, Poitras (2000) has shown that, during the 15th and 16th centuries, it became common in the algebra books to use problem of compound interest to solve algebraical problems and, some times, the good rule was given, some times not. This was a great change since the first vernacular algebra manuscripts[124]. For instance, in a 1395 manuscript quoted by Franci & Rigatelli (1988), one can find the following problem

> *A man loaned 100 lire to another and after three years he gives him 150 for the principal and interest at annual compound interest. I ask you at what rate was the lira loaned per month ?*

---

[120]The harmonic series is :

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{n} + \cdots$$

To show that this series diverge, from $1/2$ excluded, Oresme gather 2^i terms, $i = 1, 2 \ldots$ in such a way that :

$$S = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right) \cdots$$

in as in all collection, each term is greater than the last, he founds that:

S ¿ 1 + 1 $\frac{}{2+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right)+\cdots=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots=\infty}$ So Oresme demonstrated that there are series which are convergent and other that diverge. Since Oresme's time, Kifowit & Stamps (2006) has counted 20 distinct demonstrations of the divergence of the harmonic series.

[121]Just before Oresme's work, Swineshead — also known as Suiseth or *the calculator* — found in his *Liber calculationum* that the infinite sum of the series of reason $n/2^n$ equal 2.

[122]One must also note that the *exponential growth* appears in the *Shahnameh*, an epic poem written by the Persian poet Ferdowsi between 977 and 1010 AD about the game of chess — see Macdonell (1898). It's the famous grain problem linked to the prize asked by the inventor of the game of chess as his bounty. Fibonacci necessarily knew about this exponential growth since he studies this anecdote at the chapter 12 of *Liber Abbaci*.

[123]To be perfectly correct, one must note with Struik in his preface to the *Principal Works of Simon Stevin* — see Stevin (1958) —, that there exist cuneiform tablets — preserved in the Louvre — with compound interest problems which, for instance, asked how long it takes for a sum of money to double itself at 20% of interest which lead to the resolution of the equation $(1.2)^x = 2$ which was solved by linear interpolation as underline by Archibald (1949 (supplement)), which is equivalent to have an irrational power of size approximatively equal to 3.8. In some tablets, preserved in Istanbul, one may find tables of c^n for $n = 1$ to 10 which were utilized to solve equations of the type $a^x = b$ in sexagesimal notations.

[124]Since Van Egmond (1980) has published a catalogue of italian abacus manuscript and proclaimed to have found the oldest vernacular treatment of abacus — the *Libro di Ragioni* of Paolo Gerardi dated of 1328 — see Van Egmond (1978) — and that Høyrup (2007) has translated the *Tractatus of Jacopo da Firenze* — written in Montpellier in 1307, and that Franci (2010) has shown that at least two other manuscripts may be dated of the same period — *a Tractato dell'Arismetricha* written at about 1320 in Pisa which can be found in the folios lr-7lv of the manuscript 2252 of the Biblioteca Riccardiano in Florence and the manuscript 1754 of the *Biblioteco Statale* of *Lucca* which can be dated at about 1330 , the controversy rages amongst scholar since Franci (2010) and Oaks (2009) have casted doubt on the dating. One must note that Franci (2010) and Oaks (2009) have shown that those manuscript are from distinct origin without borrowing to Fibonacci's work.

which conduct to the cubic equation :

$$x^3 + 60x^2 + 1200x = 4000$$

We can see that the terminal date is perfectly arbitrary and could have given birth to any order equation. With Nicolo Fontana — better known under his nickname Tartaglia, the *stammerer* —, in 1556, for the first time, an equation of 9th order appeared through a problem asked to him by a gentleman of Bari :

> *A merchant gave a university 2814 ducats on the understanding that he was to be paid back 618 ducats a year for nine years, at the end of which the 2814 ducats should be considered as paid. What compound interest was he getting on his money?*

We must wait until Dary (2010) — published in 1677 —, to see the first analytical solution of what we now call the *implicit yield of an annuity* will generate an undetermined power equation whose solution is the sought interest rate — see also Hawawini & Vera (1980).

In the 17th century, Great Britain has known an explosion of curiosity toward calculations of whichever type of interest calculation which accrued to mathematical knowledge, it must also be acknowledge that there was a strong impetus from the market to study many sort of loans — see Lewin (1981). First off all, the logarithm tables of Jost Bürgi, John Napier and Henri Briggs[125] which give birth from its translation from latin by Edward Wright to an anonymous appendix — most probably written by William Oughtred — containing the first table of what we now call the natural logarithm.

If, in the 14th century, Oresme was the first to realize that the cubic root of x could be advantageously written as $x^{1/3}$ which lead to fractional power if, for instance, we need to square it, it was not until the beginning of the 17th century that it became mandatory to have a way to evaluate it to a high order of approximation. This was linked to a change of perspective. If Witt had shown how to evaluate to capitalization factor — $(1 + i)^n$ — for less than annual compound frequencies, he was postulating the knowledge of the smallest period rate of interest — that is to say that if i is the quarterly interest rate $(1 + i)^{T/4}$ will be the interest factor for the first quarter if $T = 1$, the second, for $T = 2$, the third, for $T = 3$ and the year for $T = 4$, it appears that this was no more of universal usage since the main shape in which interest was announced was under an annual basis[126]. So now the problem was to evaluate $(1 + (i/n))^{ni}$. This motivate to find a way to evaluate the formula for an irrational power.

From our time, its more than obvious that the solution should comme from the binomial theorem[127]. But in those times, it was not so obvious and it is because he was obliged to manage is time as withdrawn due to the plague which raged in Cambridge, that Isaac Newton finally shown that the formula could be expanded[128] for any value of i/n as long as $-1 < i/n < 1$.

Before to go further we must signal a not so known connection between interest calculations and modern mathematics. In 1606, Galileo dedicated his first work to the description of the practical applications of the new compass which he had just invented — see Galilei (1978) and Galilei (1978). He

[125] Roegel (2010) explains how, in the 15th-16th century Johannes Werner, observing that some trigonometric formulas transform a multiplication in addition, invented the *prosthaphæric* method which came to the knowledge of Tycho Brahe and Paul Wittich whose trichonometric manual began its diffusion until it touched Bürgi who improved it.

[126] This was the approach followed by Witt (1613).

[127] In the 4th century B.C., Euclid known the theorem for $n = 2$ as the indian mathematician Pingala a century later. But, in what concerns the integer power the general formula, known as the binomial theorem, was independently discovered in the 10th century by the indian mathematician Haladuya persian Al-Karaji and the persian Al-Kharaji. In the 11th, ignoring the Al-Kharaji contribution found again the formula. In the 13th century, this time round the chinese mathematician Yang Hui displayed the formula. But, to aour knowledge, Al-Kharaji was the lone to give the demonstration using mathematical induction. Then Pascal published his triangle — the text has been integrally incorporated in Smith (1929). We must also note that Whiteside (1961) has shown that Henry Briggs anticipated Newton of 50 years.

[128] Faithful to his habit to publish nothing, Newton demonstrated the theorem between 1664 and 1665. but its only in 1776 that he decides to write a letter — the *Letter prior* — to Henry Oldenburg then secretary of the Royal Society of London to transmit it to Leibniz who asked for further details on the demonstration. This gives birth to a second letter — the *Letter Posterior*. Finally, he let Wallis (1696) edit his work.

announced that his compass was able to assist in military calculations *in various ways, like determining the proper amount of charge for any size of cannon, measuring calibres, making territorial and architectural surveys, and it could be used in civil affairs, like computing compound interest and monetary exchange rates*. As Sasaki (2004) underlines, the compass was also able to extract square roots, and discover the mean proportional[129]. This would have been another curiosity, if only Renée Descartes has not had Isaac Beeckman as professor during his sojourn in Breda — Netherland. The later introduced him to this little opuscule. What seems incredible to Descartes was the fact that the proportional compass could be used to solve both arithmetical and geometrical problems. This was suggesting that these two mathematical fields could be unified. As the principle hidden behind those applications was the one of continual proportions, there was certainly a meta-discipline that could encompass both and finally it identified it to the *Algebra*. According Gaukroger (2007), not only Decartes conceived arithmetic and geometry as particular algebra species of it, but algebra would be also a powerful problem solver. And even if Descartes was more physics oriented than any other thing, he does not neglect the influence of the compound interests calculus by no mean, as is attested by a brief allusion to the subject in Beekman's journal — see Beekman (1939) — and two other rapid allusions, but that time directly by Descartes, the first time in the paper AT X,78[130] where it is the acceleration of the interest which is invoked and then in the *Cogitationes Privatae* — see Adam & Tannery (1910), vol. X. According to the position of Descartes in the history of the mathematic thought, it is amazing to observe that the interest calculation was not absence of its genesis.

Now, contrary to the vision that we have inherited from Newton, his mathematical researches were not only motivated by his the natural philosophy. He was also involved in the great questions of his time and those questions were financial questions. For instance, as soon as 1664, he constructed the *curva logarithmica* geometrically to depict the exponential growth of capital when its compounded interest accrues instantaneously from moment to moment — see Newton (1967) and Newton (1981) — and twenty years later, he used his knowledge to construct some *Tables for Renewing and Purchasing of the Leases of Cathedral-Churches and Colleges, According to Several Rates of Interest*. When, in 1707, William Whinston[131] will have obtained the authorization to publish its courses from 1673 till 1683, under the generic title *Universal Arithmetick: Or A Treatease of Arithmetical Composition and Resolution*, it will be visible that the financial model was an actual basis of Newton's reflection.

Newton was not alone of his kind. It was the same for James Gregory who, in the toward the end of 1770 years and independently of Newton, moved near the binomial theorem, what he exposed in a letter to Collins — see Gregory (1939). He realized immediately what he has in hand, since his first application was to solve a compound interest problem — see González-Velasco (2011) for more details.

If only the most part of Thomas Harriot writings had been published on his life time or soon after his passing, he should have been recognized as the first in the continuous time compounding calculus. Unfortunately, on this subject its papers remained unknown till dug up by the *Cultural Heritage Online — ECHO — initiative* and by that a very astonishing manuscript[132] be exploited by Biggs (2013). Note that since Harriot, born in 1560, passed in 1621, the manuscript could have been written more than sixty years before the contribution of Newton…But, as in his time, the mathematics necessary to write properly the asymptotic of the compound interest factor was not yet developed, Harriot. Supported by Pascal's triangle, he was able to approximate a $b^2 = 100£$ loan to $b = 10\%$ of interest for 7 years, slowly going from one period capitalization to more frequent periods according to the formula: $b^2(1+1/nb)^{n7}$. Harriot correctly inferred that, due to the presence of power of 10 in the binomial expansion of the formula the yield could not increase unboundedly. Astonishingly, his approximation was truly remarkable.

[129]The positive number x such that for two positive numbers a and b, $a/x = x/b$, that is $x = \sqrt{a \times b}$.

[130]AT X,78 is the standard citation of Adam & Tannery (1910) vol. X, page 78.

[131]Since 1701, William Whinston was the substitute then the successor of Isaac Newton to the Lucasian chair of mathematics. in Cambridge

[132]Manuscript BL Add Ms 6782 f. 67 © British Library Board.

Nevertheless, the very simple formula for the continuous time compounding was yet to come. According to many sources, it came in 1683, through the Jacob Bernoulli search of the limit of $(1+1/n)^n$ that the 100% interest factor. Groping, he found that the limit was higher than 2.5 and less than 3 But he was unable to linked this value to the logarithm. Later, in his *De seriebus infinitis* published in 1689 — see Bernoulli (1713) —, discussing a problem of continuous compounding he find a series of the form:

$$1+\frac{x}{s}+\frac{x^2}{1.2.s^2}+\frac{x^3}{1.2.3.s^3}+\frac{x^4}{1.2.3.4.s^3}+\cdots$$

without identifying to the inverse of the hyperbolic logarithm even if in an earlier of his writings he has noted that the exponential series can be regarded as the inverse of the logarithmic series. The Bernoulli result[133] was found independently by Euler who, once more time, justified his research through the compounding problem. Euler gives the name e to the constant in a letter to Goldbach of November 25, 1731[134]. In 1748, he was able to give an approximation of e to 18 decimals without giving any explanation to the method he used[135] — see Euler (1988).

Those results, even if they were compound interest results and not a present value results, but the formula's inversion is so christal clear — seems to have sounded the knell of any serious research on continuous time finance, at least for nearly two centuries. It is surprising all the more, since Euler was particularly involved in the calculus or the *rentes viagères*. According to Sandifer (2008), we can award him with at least six papers on the subject and a dozen of others on the probability associated with the lotteries where in card games. But, and this is the point, Euler was the first to investigate the transformation later known as the *Laplace transform* which is no more than the present value of a perpetual annuity function $f(t)$ — *i.e.* :

$$\mathcal{L}(r)=\int_0^\infty e^{-rt}f(t)dt$$

Unfortunately, even if mainly developed in the first quarter of the 19th century, it is only in the second part of the 20th century, that social scientist have realized the link between Laplace transform and present value of a perpetual stream of income — see Grubbström (1967) and Grubbström & Yinzhong (1989)[136]. Of course, present value is a Laplace transform only for a perpetual stream of income, but it is astonishing that the analogy should have not spread.

There are at least two main reasons that in Euler's time the impetus for the mathematical research leave the finance research: first, the effective speech in defense and activism of Voltaire in favour of the Newton's natural philosophy leads to refocus the efforts of the mathematicians on the physicians' specific problems[137]. Then, in spite of all the efforts of the financial theorists, the life insurance domain set aside, until the last quarter of the 20th century, the echoes of the theory at the practitioners remained very weak. Ward (2009), in the beginning of the 18th century, was not the first one to complain about it. There was a profound reason about it: finally, the simple interest had gained the favors of churches, and appeared, more or less, as the only justifiable practice[138]. In fact, since roman times, we know that simple interest is a practice that can be easily transformed in compound interest. Suppose we want to target a compound interest rate r but only simple interest i is legal, since $1+it$ is the value of the capital augmented of the

[133] According to Poitras (2000), Pearson (1978) credits Halley for the proof that $\exp(x)$ is the limit of $(1+(1/n))^{nx}$ when n tends to infinity. What is very confusing is that the credit to Bernoulli comes always with no references at all. There is no references in Coolidge (1950) who cites Euler but make no mentions of Bernoulli.

[134] In fact, Euler used it earlier between 1727 and 1728 in an unpublished paper on explosive forces in cannons — see Smith (1929). So the story of the choice of this letter, against the b that Leibniz use in two letters to Hyugens — see Mitchell & Strain (1936) for the use of other letters and symbols —, to pay him tribute, is for most an *ex-post* rationalization.

[135] If we use the former formula with 20 terms, we can find exactly Euler's approximation.

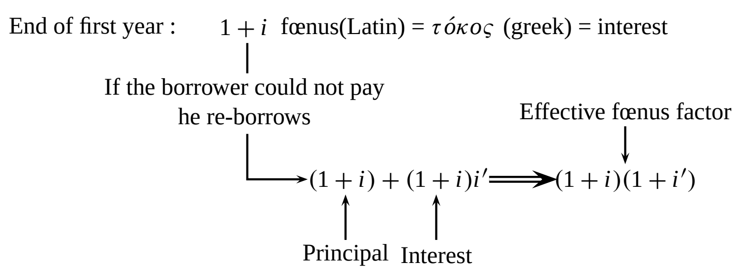
[136] See also Buser (1986).

[137] Its under Voltaire's influence that Émilie du Chatelet has translated in French, the *Principia* of Newton — voir Newton (1759).

[138] We know now that it is profoundly inefficient and inequitable to the lender.

accrued interest under simple interest for a length of time of t years, when $(1+r)^t$ is the formula for the compound interest. Equating this two formula and targeting for a certain value of r, we can find the equivalent simple interest rate. For instant to 7 years, for a target of 15%, we find that the simple interest rate is around 23%.

According to Geisst (2013), which takes his information from Smith, Wayte & Marindin (1890), in fact even under simple interest regulatory laws, compound interest was always present under a proceeding which was known since the latin time as a *versura*[139]: if a debtor could not pay the principal and interest at the end of the year, he used to borrow money from a fresh creditor to pay off his old debt which gives the sequence :



As Romans had already realized that it was possible to release themselves, in a doubtful way from the obligation to repay their debt by organizing their insolvency, it was the so called case of *alienatio in fraudem creditorum facta*. According to Smith, Wayte & Marindin (1890), when a man was insolvent and alienated his property for the purpose of injuring his creditors, the prætors gave him some remedies. For instance, if a debt was paid *post bona possessa*, it declared was void, to put all the creditors on an equal footing. And if the borrower act in such a way to diminish the value of his property, his act could be undone. Romans and modern jurists spoked and peak of the *actio puliana*. This was necessarily known since the re-discovery of the *Justinian's Pandectes*.

Whichever be the resonance of the questions which have been raised, they leave without answer the fundamental question to know who the first gave the quasi-obvious formula of the life annuity. So far as our investigations allowed us to go back up, it seems that the first to bypass the barrier of the life expectancy seems to have been Leibniz (1683)[140].

The Leibniz problem was to determinate the *rabat* value, that is to say the difference between the future value of a financial claim ant its present value, a problem which arise when a sale involves an immediate cash payment which will be later completed by a series of payments to terms spread out in time. So the Leibniz question was to know when the seller has to require an immediate payment or accept its spreading[141]? If, he was following the legal rules which in his time and saxony country, that is simple interest, he would find to some stupid results. Finally, Leibniz moved forward the following arguments : suppose that the creditor receives a sum S, one year before his normal term. After a year, he will owe an

[139] They also speak of *usurae renovate*.

[140] Leibniz has written 50 texts — letters and manuscripts — dedicated to financial mathematics which have all been translated in german in Leibniz (2000). We must notice that he has only published the 1683 paper. They are analyzed in two french books — Rohrbasser & Véron (2001) and Parmentier (1995) — and one article — Knobloch (2006).

[141] Leibniz began to destruct the accepted motto of the middle of 17th century which has been enounced by the famous saxonian jurist Benedict Carpzov who claimed that the rebate had to be calculated on the basis of the interest on the money which the buyer had not yet paid at the beginning of each year — for a description of Carpzov's method see Neuburger (2000). In Leibniz own words, *the interest on the outstanding payments could be higher than the money paid in cash. In such a case the bidder had paid less than nothing* — see Leibniz (2000). The strength of his arguments and the authority which conferred him(her) during more than a century the co-invention of the mathematical analysis, led to forget completely that he was not the inventor of the present value as shown by Oettinger (1861).

interest of iS. But if he accepts to immediately subtract this interest from the sum he will receive, now it is the lender who must repay him an interest of $i(iS)$ that is $i^2\ S$. If the lender accept to immediately pay this interest, the creditor will receive $S - iS + i^2\ S = S(1 - i + i^2)$. Extending this step by step reasoning, he finally arrive to define the present value a a sum S available in one period of time as[142] :

$$V = \left(\sum_{k=0}^{\infty}(-1)^k i^k\right) S = \frac{S}{1+i}$$

This approach, although ignored by finance textbooks, not only major ones but all, is a major step in the *anatocism*[143] theory, because Leibniz doesn't follow Fibonacci's road[144], since in his time Fibonacci has been completely forgotten. What he shows, is that present value under compound interest can be understood as a linear nexus of interdependent loans where, by turns, lenders become loaners and vice-versa never departing from a simple interest base. This formula being true for a year, it could be generalized to any length of time. So Leibniz reinvents the present value from a completely new ground since Fibonacci does it nearly five hundred years earlier. As, when possible, there are only two way to expose a continuous function — directly through its expression or through its series development, four centuries after the breaking Fibonacci's thought, the theory of the discrete time present value was complete. For Leibniz, the value of the pension emerges from an incredible trade between two typify persons whose temporality is immense: the person of private means who, by nature is mortal, and his offspring which is immortal— as far as peoples always have at least a heir, if only the State. What distinguishes essentially the person of private means is that if he was immortal, the rent payments would run eternally and all his offspring would beneficiate of this. But, as a mortal, he renounces to transmit this eternal enjoyment to be the own beneficiary during his life. So, he has to pay to collect, in an anticipated way, the pensions during his life time and, what he pays is what Leibniz call the *rabat*[145]. Unfortunately, in spite of the very will of Leibniz himself to promote its reasoning — will that is certain since it is the only result he obtains in the matter that has been published during his living time, even known from many authors, his incredible strong achievement stay unnoticed even in our time.

Of course, in the middle of the 17th century europe, the great debate was questioning life insurance. The earliest life insurance contract of the modern era was bought in 1583, by a London salter, Mr William Gybbons to benefit to a certain Richard Martin for a one year term. When Gybbons died less than a year after the contract the insurer refused to pay under the strange argument that the contract was running on a lunar year[146]. The affair ended in front of the court which refused to follow the insurer who was obliged to pay.

This certainly influenced the search for a rationalization of the tariffs and, when Graunt (1662) published the first mortality tables, demonstration was done that there was room for a scientific way to price. In those time the essential work of Halley (1693) and de Moivre (1718) was yet to come, and Leibniz was certainly one, if not the first to realize that life annuity was amenable to a present value calculus throughout the statistical evaluation of life expectancies. And early was he involved with insurance, specifically life insurance. He his certainly one of the first author in the modern time to plead for the establishment of a life insurance. Approximately from 1678 to the end of the 17th century, he wrote several texts about

[142] The resolution comes simply from the fact that one may realize that : $V + iV = 1$.

[143] In itself the roots of this word which is a synonym of compound interest speaks by itself: in greek, *ana* means one more time and *tokos* means income.

[144] Leibniz was not thinking that he was advancing some breaking the law. He writes to his friend Christoph Pfautz that his proposal was not concerning anoticism because the calculus take into account future virtual interest not an before term interest without counterpart.

[145] Leibniz being a particularly practical spirit, in practice he does not conceive an infinite horizon but limited his calculations to 300 years. This changes nothing the numerical values he obtains but modifies profoundly the theoretical analysis — see Rohrbasser & Véron (2001).

[146] A lunar year is 345 days long.

this subject, both on the theoretical plan tan at the application level[147],[148]. But to the exception already noticed, these texts stayed unpublished and the impact of his thoughts stay less than marginal. Nevertheless, the conception that life expectancy could be, with the good instruments, amenable to a statistical treatment, make the field of life insurance a place of choice to apply present value.

It was not the same in the other fields. To be able to apply present value, it was necessary to be able to apply probability calculus and its empirical counterpart : statistics. Even if we can list the advanced on the two subjects all along the 17th century, the Laplace (1812) and we must wait more than a century to see the apparition of the first books helping to learn practical statistics.

While waiting for innovations, according to Deringer (2013), peoples uses at least one other method to evaluate an income generating device. First, disposing only of the current income, they applied a multiplicative coefficient called the *years' purchase* which, in a certain way, should have evaluated the bid-ask spread on the specific item. For instance, if a land generated a 100£ income and there was a 20 years' purchase coefficient, the exchange value of the land was 20,000£. Obviously, there was no such thing as a market evaluation of the years' purchase. We cannot really speak about determination but rather about a convention ore even of an heuristic — see Habakkuk (1952) and Clay (1974) for a reconstruction of the way seventeenth investors used the year's of purchase.

According to Parker (1968), outside the financial area, there was certainly no application of present value until the nineteenth century. One of the main reason was the lack of huge real investment. But with the emergence of railroads, things were going to change, because to expect the least return of a rail road it is necessary to mobilize huge funds. The difficulty with railways was — and always is — that new facilities are not build on existing traffic but by projection of future traffic. Without any mean to estimate accurately the future traffic, if was very difficult to balance between audacious and conservatives expectations. Its the reason why projections were computed between short and mean terms, that is to say, between 3 and 10 years, with conventional rate of growth because observations were rare and it was not obvious that growth be sustainable — see for instance Wellington (1887). A discussion on present value was also led to decide the replacement of machines by more efficient ones — see Pennell (1914). The difficulty to evaluate non-stationary cash flows, despite the efficient plaid of Fisher (1907) extended in the famous Fisher (1930) toward the use of the net present value, leads Williams (1938) and, later Gordon & Shapiro (1956), to develop against the Keynes' vision of the stock market as a casino, a model of the value of the firm as the present value of the future dividends in a constant growth context, that is to say that, if P_0 is the present value of the firm, D_1 its current dividends and g its rate of growth :

$$P_0 = \sum_{t=1}^{\infty} \frac{D_1(1+g)^{t-1}}{(1+i)^t} = \frac{D_1}{i-g}$$

Unfortunately, during the 1950's years, Clendenin & Van Cleave (1954) and then Durand (1959) showed that there was a draw back with a constant growth rate[149] since for $D_1 > 0$:

$$P_0 = \{ > 0 if\ g < i \infty if\ g = i$$

So the present value be infinite. And, according to Durand (1959), its not the lone problem. There is no reason that the firm be always growing. At any time, it can peter out, as in the *Petersburg paradox* where at any time you can win a lot of value 2^k — in the firm case, loose — with probability 1/2,

[147] For more details see Graf von der Schulenburg & Tomann (2010).

[148] Amazingly, its because he passed the information send to him by Casper Neumann to Justell, then Secretary of the Royal Society who in turn passed them to Edmund Halley. than this later was able to publish his 1693 paper.

[149] Notice that if a firm decide to retain all its earning and not distribute dividends, D_1 could and must be interpreted as the *book value* of the firm, that is its actual cash value or to its acquisition value, that is to say its cash value augmented of certain costs tied to the purchase of the asset.

provided that you have not win yet[150] in the kth throw of a coin and, because of that has an infinite mathematical expectation. Amazingly, the analogy between present value and the expectation in the *Petersburg paradox*, leads Durand to propose for the infinite case to adopt solutions proposed to solve the paradox, mainly to shorten the horizon into a finite one.

We can legally wonder about the origins of place taken by an analysis of the present value based on a dividends constant growth rate. A simple reason is that before 1960s, the methods of prediction / projection of the cash-flow, which can only be based on a developed time series analysis, were still very confidential and were usable only by the researchers. Until the first quarter of the 20th century, forecasts were based on the presupposed idea of an subownerless invariant strictly stationary world. That is to say, in the case of cash-flow prediction, that is to say that, the common design claimed that cash-flow were only realizations of a single random variable. This lead to its expected present value calculated as the product of the expected cash-flow and the present value of one unit of account[151] — *i.e.* :

$$P_0 = \mathbb{E}\sum_{t=1}^{\infty}\frac{\tilde{D}}{(1+i)^t} = \sum_{t=1}^{\infty}\frac{\mathbb{E}\tilde{D}}{(1+i)^t} = (\mathbb{E}\tilde{D})\left(\sum_{t=1}^{\infty}\frac{1}{(1+i)^t}\right)$$

Of course, it was possible to take into account a random rate of growth of the firm, provided that the latter can be considered as strictly stationary. With the emergency of *time series analysis technics* in the second half of the 20th century, things changed and the strict stationarity was no more a constraint in such a way that the cash-flow could be describe in a more sophisticated way.

Even if one can date the beginning of the time series theory to the series of Fourier (1822) and the first appearance of the correlogram with Schuster (1906), without forgetting the work of Buys-Ballot (1847), its really with the introduction of *autoregressive models* by Yule (1927), the *moving average models* by Slutsky (1937)[152], and the unification of both approach through the so called *Wold theorem*[153] — see Wold (1954)[154]. Since then, the analysis of time series has been extended to a huge lot of different observed series and give room to the estimation of cash-flow, provided that we accept the idea that the future is essentially engendered in the past.

An other path, which is today at the core of the usage of the present value in finance has been initiated by Bachelier (1900)[155]. Unfortunately his work spread very slowly amongst essentially probabilists until the beginning of the 1960. Then when all the mathematical instruments — essentially when the tool box of stochastic differential equation, developed mainly by Ito (1944) and Itö & McKean Jr (1964) — reached a sufficient degree of maturity, Black & Scholes (1973) and Merton (1973) in the same time where able to construct a satisfactory theory of the price of derivative assets supported by the *arbitrage notion* borrowed to Modigliani & Miller (1958).

The basic idea is that, in an economy where all investors are risk neutral, its very easy to price a derivative, that is an asset which give a remuneration based on a specified stochastic process, according to the fact the, if it exists a non risky asset of today price V, this one has, in continuous time, a value of $(1+rdt)V$. And, as risk neutral investors decide always according to the expected value of a risky asset, for an investor who invest the same amount of money V in the risky asset than in the non-risky one, the former must have the same final value. As this final value come either from capital gains — dV — and from the remuneration — π — distributed by the asset, one must have :

[150] In the firm case, postulating a 1/2 probability of failure, consists in calling Laplace's insufficient reason principe, which set to 1/2 the probability of any event about whom we know nothing.

[151] Where $\tilde{D}$ is the cash-flow interpreted as a random variable, and $\boldsymbol{E}$ is the expectation operator.

[152] As the Yule paper, Slutsky's one was written and written in russian in 1927. It has been translated and published in english only ten years later.

[153] Wold (1954) enounce the conditions under which every covariance-stationary time series X_t can be written as the sum of two time series one deterministic and the other completely nondeterministic.

[154] Although already dated, a good survey of the beginning of the time series modeling approach of real data is given by Makridakis (1976).

[155] See Jovanovic (2009) for a extended historical treatment of the fate of Bachelier.

$$(1 + i\,dt)V = \mathbb{E}_0\left[dV + \pi(X)\right]$$

With the additional hypothesis that $X(t)$ is described by a non-standard brownian motion, this gave birth to an equation which solution can be given through the formula of Feynmann-Kac[156], which is no more than the expectation of a present value of the income flux augmented of the terminal value of the asset — that is:

$$V = \int_0^T e^{-it}\pi(X(t))dt + e^{-iT}\Omega(X(T))$$

where $\Omega(X(T))$ is the terminal value of the asset. Of course, one may wonder why there is no references to risk preferences. It's because they are already incorporated if $X(t)$ is the price of a marketed asset which has been valued according in a perfectly competitive market.

One must notice with Deringer (2013), that compound interest finally killed nearly every other way to evaluate an income flux during the first half of the 18th century. In the post *Great Fire*[157] years London's housing market, it was already present during the allocation of grounds for the reconstruction as documented by historian as Baer (2002), Glaisyer (2007) and Harrison (2001) to cite only a few. In a certain way, it was strange since many peoples have already observed that with compound interest the price of annuities was inversely related to the interest, which means that landowner had interest to drive it to the lowest possible value.

One of the first plaid to convincingly argue in favour for the use of compound interest and present value was a member of the british parliament, Abraham Hutcheson who, in three pamphlets, denounced, before it failed in the end of 1720, the commercial prospects of the South Sea Company, which was a quasi-public trading company proposing a very tempting scheme to refinance Britain's national debt, precipitated the country in its first great financial bubble — see Hutcheson (1718), Hutcheson (1720) and Hutcheson (1721). In his 1720 firebrand, he treated the stock as a seven years fixed term annuity through an exponential discounting, in such a way to calculate the trading profit the company could realize to support dividends that justified its exorbitant stock price. It was a complete new approach to this subject, which was approached by other pamphleteers in the more customary political way following the celebrated book of Defoe (1701). As a history wink, in spite of his final impact on the financial customs of his time, Paul (2007) has shown that Hutcheson's arguments were poor economics and was *hidebound by traditional assumptions about the importance of the landed elite*.

The second plaid was implicit and revealed by the exponential number of english books dedicated or given a huge place to *exponential present value* — see for a large sample of those books Lewin (1970). The main part of those books present both simple and compound interest rules, which demonstrates that if simple interest was yet the common rule, the authors could not avoid presenting the compound interest rules compound interests either to convince their readerships or because the latter were eager to learn how to use them, what on the whole returns to the same.

And finally the game was made when in 1716, Robert Walpole, then chancellor of the Exchequer, proposed a new way to manage the then ever-growing national debt, developing his arguments on the basis of compound interest. He promoted the new idea of the *sinking fund*[158], borrowed to the end of the 17th century pensionary of Holland, Johan de Witt, who incidently has written the first modern treatise on annuity, was to use the power of exponential growth in reverse — see Brisco (1907). He reused this fund by two times in the 1720s and early in the 1730.

[156] See Feynmann (1948) and Kac (1949) and, for instance Neftci (2000) for a financial presentation.
[157] 1666.
[158] In a *sinking fund*, a part of the annual incomes is set aside to repay a long-term debt and, eventually, some futur capital expenses.

What is for us the most astonishing observation, is that there were no real argument in favor of the compound interest against simple interest. For instance, alongside with Fibonacci's and Leibniz's arguments in favour of compound interest, one would expect to see a temporal arbitrage argument defending that compounding interest is the only way to balance fairly between short and long terme financing. Indeed, if the i is the constant short term interest rate and R the two term long interest rate, owing for the legitimate reinvestment of the interest, one must have :

$$\underbrace{1+R}_{longterminterestfactor} = 1 + \underbrace{i}_{\text{First period interest}} + \underbrace{(1+1)i}_{\text{Second period interest on interests}} = (1+i)^2$$

So clearly, in a world of certainty, if $1 + R > (1 + i)^2$, investors will never invest in short terms projects. Because, if we suppose that compound interest is strictly forbidden, the long term interest rate will be fixed in such a way that :

$$1 + R = 1 + 2i$$

but, in that case, it is self-evident that $R = 2i$ is too small in comparison with the required $R = 2i + i^2$.

Astonishingly, it is in the traumatic pre-revolutionary France that the first alternative criterium to the present value will appear under the feather of Duvillard de Durand (1787). To explain how Duvillard's criterium has been constructed, it is necessary to jump a century and a half later to *The General Theory of Employment, Interest, and Money* de Keynes where he defines the *Internal Rate of Return* of an investment as the interest rate that will cause the sum of the discounted future cash flows — the series of CF_k, $n = 1, \ldots, n$ —, exactly equal to the initial cost of this investment — I_0. If $\mathbf{i}$ is this rate, it is then defined as a root of[159]

$$\sum_{k=0}^{n} \left(\frac{CF_k}{1+\mathbf{i}}\right)^k - I_0 = 0$$

Now, following the interpretation of Biondi (2006), in the case of annuities, which is the Duvillard case, this expression is simply :

$$\sum_{k=1}^{n} \left(\frac{C}{1+\mathbf{i}}\right)^k = I_0$$

for C constant.

Here Duvillard was not used to work with the *Net Present Value* but with the *Net Future Values* — *NFV* — and one knows that the formula $NFV = (1+i)^n \times NFV$, let go from the former to the later. Thus one can also write :

$$C \times \sum_{k=1}^{n} (1+\mathbf{i})^{n-k} = (1+\mathbf{i})^n \times I_0$$

and as the first factor can be written as $((1 - \mathbf{i})^n - 1)/\mathbf{i}$, the former expression is simply :

[159] Despite the fact that any management student has been a introduce to the Net Present Value and to the Internal Rate of Return, a recent study by Ryan & Ryan (2002) indicated that on the *Fortune 1000* companies, that is the 1000 larger american companies ranked by sales, 49.5% of them use always NPV and 44.6 % always use IRR. Of course, this result is disrupted by the fact that some companies oscillate between criteria. When one take into account also those which some time use those criterion, one find that the usage of the NPV is of 92.1 % of the correspondant and the usage of the IRR of 76.7 %.

$$C \times \left(\frac{(1-\mathbf{i})^n - 1}{\mathbf{i}}\right) = (1+\mathbf{i})^n \times I_0$$

If we stay there, we are simply dealing with a particular expression of the internal rate of return. But what Duvillard clearly understood, and this will be rediscover only nearly 160 years later by Lorie & Savage (1955), is that there is an unrealistic point in the formula : the cash-flow are reinvested at the rate of return of the project that generate them, which give an unduly optimistic value for the IRR. In reality, the rate of interest to whom they must be discounted is more correctly captured by the cost of capital, that is the rate of return that the same amount of capital could be expected to earn in an alternative investment of equivalent risk. In that case, if δ is the known cost of capital the *Modified Internal Rate of Return* will be solution of the equation :

$$C \times \left(\frac{(1-\delta)^n - 1}{\delta}\right) = (1+\mathbf{i})^n \times I_0$$

and as the left member is a constant, say m, the *MIRR* is given by

$$\mathbf{i} = -1 + \left[\frac{m}{I_0}\right]^n$$

Now to chose between many project is simply to compare the *MIRR* to the rate of interest and chose in decreasing order the projects which give an *MIRR* greater than the interest rate.

Here of particular interesting is the fact that Duvillard found the *MIRR* largely before than Keynes introduced the *IRR*. Unfortunately, for many reasons, among which some bound to his personality, and other to his former links with Condorcet who supported the publication of his work[160] — see among other Biondi (2006), Biondi (2003) but also Baumol & Goldfeld (1968).

Since up to Baumol & Goldfeld (1968), Duvillard was nearly forgotten, it is around Keynes' *IRR* that the concurrence with the *NPV* was developed. Of course, we must have been aware that the criterion is implicit in Boehm-Bawerk (1889), where its is taken for *granted that businessmen would (and should) invest so as to obtain the greatest annual net cash flow in perpetuity per dollar invested — a simple version of the internal rate of return principle*. Since the middle of the 20th century, the limits of the NPV has been largely publicized and extended: Hirshleifer (1958) has explored the impact of market imperfections on the NPV, Marglin (1963*b*) and Marglin (1963*a*) have shown how to incorporate the possibility of reinvesting in the criterium, Wright (1959), Flemming & Wright (1971) and Arrow & Levhari (1959) have studied the effect of the choice of the duration of the project on the present value. Recently, Ross (1995) has shown that the present value can reject an investment when it should be accepted and, unfortunately, accept an investment who must be rejected, because it doesn't take in account the optionality which is incorporated into one project and arise since the project compete always with himself when delayed in time. Overall, the main reason why the use of the IRR as the investment decision criterion is that it may not be unique[161]. Obviously, more than 800 years after its creation, Fibonacci's NPV stays, the main criterion to evaluate a project.

Let us return to Leibniz. In 1690, Locke published his famous *Essai on Humane Understanding* — see Locke (1690). As Patterson (1967) put it, the book *as done much to shape the course of intellectual development, . . . , ever since it has been published in 1690*[162].

[160] As explained by Gallais-Hamonno & Rietsch (2003), Duvillard never obtained to be elected as *professor of social mathematics* at the *Collège de France*. He used to attribute his two times failure to the hostility of Laplace against the Condorcet's boys.

[161] One can read Rocabert, Tárrio & Pérez (2005) for a tentative to rehabilitate the IRR.

[162] It was such a selling success that, before the passing of Locke, which occurred 14 years later, it run four editions.

For what concerns us here, Locke was thinking that human beings are social creatures by nature rational an equally free. And, as argued by Iversen-Vaughn (1980), this rationality unifies the Locke's economic thought, through it's theory of social institutions formation — known as the *theory of consent*[163] — see Woodhouse (1938).

In the first edition of the *Essai*, Locke asserts the agents evaluative judgments — the preferences — determine their conative attitudes and also their attitudes in front of certain goods and/or bads. In other words, if we judge that R_1 is preferable to R_2 and if the A_i act involves the result R_i — $i = 1, 2$ — then, if we are prepared to act immediately and that both act A_i are feasible, then our judgment determines that we undertake the A_1 rather than A_2. But, in 1692, the Irish philosopher and political writer, William Molyneux send a letter indicating that this involves that all the sins proceed of our understanding or are committed against our consciousness. As a consequence, a man could be damned because he does not understand better that he acts.

Locke took Molyneux' objection seriously. This will lead him to revise twice — for second and for the fourth edition — the chapter II, in such a way to avoid the consequence according to which all the troubles do not ensue from our understanding. From now on, Locke is going to try to take into account the mental dysfunction spotted by the sentence borrowed in Ovide's *Metamorphoses* "*meliora, proboque, damaged to sequor*" — *I see better things, and approve, but I follow worse*[164] —, phenomenon known as *weakness of the will* or still, *akrasia*[165]. ***

Of course Locke was not the first to make references to *time preferences* since it was implicit in the works of Saint Thomas Aquinas and explicitly formulated by his brillant disciples Aegidius de Lessinia in 1285 :

> *futures goods are not valued as highly as goods available immediately , nor are they as useful to their owners, and therefore justice dictates they should be considered les valuable* — see de Lessinia (1882)[166] .

Later San Bernardinio of Sienna, while in the thomistic tradition who forbade interest on the basis that it is a sell of time, which in itself cannot be owned, found an exception: time in a certain way the relevant period during which a particular good is used. As a specific good could be used for a specific task during a specific period, this period could be regarded as private property and therefore sold. His idea is that if someone lends his property to someone else for a specific time period, he could charge payment for such use and that payment is what we know call interest[167]. Unfortunately, he was not able to recognize the significance of his finding. Then, nearly simultaneously, Martin de Azpilcueta Navarus in Spain and Konrad Summerhart in Germany came back on the subject. Building his own thought on Bernardinio, Azpilcueta posed that

> *A claim on something is worth less than the thing itself, and . . . , it is plain that that is not usable for a year is less valuable that something of the same quality which is usable at once — see de Azpilcueta Navarro (1965).*

In his 1499 *Treatise on Contracts*, Konrad Summenhart, the great Tubingen theologian, began to admit exception to usure prohibition. He began the process of the desatanezation of money, taking on

[163]In social philosophy, *consent theory* postulates that individuals enter in consensual relationships with other individual as rational, free agents. Any gouvernance which doesn't respect this postulate as basis for political governance is then a failure. As Oliver Cromwell, a famous opponent to consent theory, set it *the consequence of this rule tends to anarchy, must end in anarchy*

[164]See Ovidius Naso (1922).

[165]Let us remind that the problem of Akrasia appears, in the 5th century B.C., in the *Protagoras*, a Platon's dialogue in which Socrates condamnes akarasia as an illogical moral concept. Half a century later, Socrates will rehabilitate the concept, by deducing that, in human being, science can be dominated by the passions.

[166]The little opuscule written by Lessinia has, for a long time, been attributed to Aquinas. Its the reason why it is incorporated in the *Opera Omnia* of St Thomas.

[167]On that subject, one can consult de Roover (1967) and Rothbard (1995).

that it is fruitful and as a good not distinct of the others fitted to exchange. He argued that in lending, a money holder is giving up something that would be otherwise profitable. Then he must be compensated for his loss, exactly in the same way as any merchant asked to receive a compensation to deliver any kind of good.

This finally results in the rehabilitation of the interest loans through a fuzzy argumentation centered on the time value, but no arguments was at the level of Locke's ones.

5 Does Fibonacci's impact the evolution of the financial markets

The question is now to know if there is a canal through which the Fibonacci legacy diffused through Europa, to finally succeed eight centuries later in a complete theory. Spiesser (2004) has tried to answer to this question. According to van Egmond (1988), there is only three *Liber Abbaci* which remains in french libraries and, according to L'Huillier (1990), at least Jean de Murs has read it in the 14th century, but even if her arguments are convincible, de Murs has not quoted it. And the principal treatises written in french or in occitan appeared only in the second part of the 15th century, most probably in reply to a demand for training because they are all educational treaties. Obviously, the links between *Liber Abbaci* and the french algorithmic treatises of the 15th century, even if they exists, are weak. So how to explain that there is no other claim to the paternity of the present value that the one of Fibonacci.

In that subject, there is no more than, conjectural

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