Can a platform make profit with consumers'mobility ? A two-sided monopoly model with random endogenous side-switching

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Abstract

We model a specific two-sided monopoly market in which agents can switch from a side to the other. We define two periods of time. In the first period, agents buy the platform services on each side and in the second period of time, they can possibly enhance their satisfaction by going to the other face of the platform. We analyze the link between mobility, consumer's utility, prices and profit. We show that mobility is a valuable feature which can be compared with an increase of product quality. Finally, the firm is able to capture the mobility in its monopoly's profit. The relative size of each group then appears as a strategical variable for the firm.

JEL Codes : L11, L13, L86
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1 Introduction

In two-sided markets (TSM), two sets of agents interact on a firm or an organization called a platform. Each group of agents generates indirect network externalities that positively (with some exceptions) affect the utility of the agents of the other group. A lot of activities are characterized by these specific cross externalities. For example, malls must attract buyers and sellers, newspapers sell its services to readers and advertisers, card payment systems must develop the merchants' acceptance in order to attract new cards holders etc.

An abundant literature has produced a fruitful analysis in several directions [1]. For example, pricing within the platform (Hagiu, 2006 ; Schmalensee, 2002), competition between platforms (Armstrong, 2006 ; Caillaud and Jullien, 2003 ; Chakravorti and Roson, 2006) and antitrust issues (Evans, 2003 ; Rysman, 2007 ; Evans and Richard Schmalensee, 2013) are important topics of TSM economics. We would like to contribute to the pricing analysis for a specific class of TSM characterised by agents' migration from one group to another one. We think that side-switching in TSM is a fruitful topic that deserves further theoretical investigation. Gazé and Vaubourg (2011) have identified this particular type of TSM.Indeed,there exist numerous examples of such TSM. In the auction website eBay, in all payment systems, in electronic marketplaces (like "Amazon Marketplace") people can achieve a transaction in one group then make another transaction in the other group. In a first time period, an agent can act as a buyer and in a second time period he can move to the other group and become e.g. a seller. These specific class of TSM raises new interesting economic questions. Do the side-switching behavior has consequences on the firm's strategy ? What impact has mobility on prices and profits ?

One potential consequence of mobility is the raise of the platform's transactions number. If agents can move quickly and easily from a group to another group (e.g with two mouse clicks), the platform can mechanically develop the market demand. For example, in the electronic market E-bay, a consumer can discover the platform from the buyer side and then have the idea and the possibility of selling second-hand objects on the sellers' side. Then, we conceive easily that mobility encourages a growth in market demand and potentially an increase of business for the firm. Actually, we are looking for more subtle mechanisms in order to have a better understanding of the "economics of mobility". We want to show that side-switching can affect prices and profit even without market growth.

In the case of a duopoly market Gazé and Vaubourg (op. cit.) demonstrate that the agent's mobility affects the equilibrium of the firms. They show that under a specific set of constraints, the firms' profit increase with the rate of the mobility of agents. The mechanism that generates profit is explained by changes in the set of prices that reflect a complex game of rewards and penalties. Firms obtain a profit from the heterogeneity of platform's consumers and from their mobility inside the TSM. Nevertheless, this analysis has a number of limitations. One of these limitations is the exogenous rate of changing face. In this paper we would like to remove this restriction ; we will show that the platform considers the mobility rates as strategic variables.

We conceive a model to highlight the effects of mobility on prices and profit of the firm which produce TSM services. We suppose that the firm is a monopoly that faces two groups of agents who can switch to the other group when their migration enhances their satisfaction. So their probabilities of departure from a group to the other are totally endogenous to the pricing of the monopoly. The monopoly has a direct impact on the size of the two groups when it modifies its prices. We show that the firm encourages agents' mobility in order to raise its profit. Our framework suggests that the monopoly has the possibility of enhancing the quality of the platform by stimulating the side-switching. In our model, promoting side-switching appears as a costless quality enhancing strategy

1. See the surveys of Roson (2005) and Rysman (2009)

for the platform. The rest of the paper is organized as follows. In Section 2, we describe the important assumptions of our model concerning consumers, firm and the timing of action. Equilibrium of period one and period two are calculated in Section 3. In Section 4, we wonder in what extent our theoretical results are relevant to have a better understanding of the actual firms' management.

2 A platform that faces mobile consumers

2.1 The consumers

The firm sells its service to two groups of agents denoted a and b. We call "consumers" agents of both sides since they consume the output of the monopoly. So this denomination can be used to describe e.g. the buyers side as well as the merchants side. Conforming to a TSM framework, the participation of one group raises the value of participating for the other group. Consumers interact through the platform and their utility is positively affected by cross network externalities.

A consumer that joins face j, $j \in \{a, b\}$ yields at period i, $i \in \{1, 2\}$ obtains the following level of utility

$$u_i^a := \alpha_a N_i^b - p_i^a,\ u_i^b := \alpha_b N_i^a - p_i^b. \tag{1}$$

We denote α_j as the cross network externality parameter and N_i^j as the number of face's j consumers at time period i. The parameter α_a (resp. α_b) measures the benefit a group-a (resp. group-b) agent enjoys from interacting with each group-b (resp. group-a) agent. Finally, p_i^j is the price charged to the consumers of group j during the i time period. We assume that consumers must buy one platform's service during each period of time. The market demand is the same in period one and two but the respective size of groups can vary from one session to the other. In addition, the total number of consumers $N > 0$ is given by

$$N := D_a(u_1^a) + D_b(u_1^b) \tag{2}$$

where, $D_j(\cdot)$ are demand functions which will be defined in the section 3.1.

We do not focus our analysis on the total size of the market but on the demand repartition on the two faces. We assume that there is no market growth from period 1 to period 2. This hypothesis means that firm is not able to earn more profit by attracting new customers on the market during the period 2. This is not realistic but it is useful to analyse and isolate the role played by mobility. In formal word, we suppose that in period 2 utility is big enough so that consumers prefer join the platform instead of staying out of the market (the market is fully served). Finally, participation's conditions are given by net consumer's expected utilities must be positive or null, i.e. : $\mathrm{E}(u_i^j) \geq 0$, $i \in \{1, 2\}$, $j \in \{a, b\}$, where $\mathrm{E}(\cdot)$ is the expectation.

2.2 The platform

We focus our analysis on the most simple market structure since we assume that only one firm faces all the consumers. We choose the monopoly paradigm in order to highlight the effects of mobility on the economic link between the firm and the consumers. Obviously, in the monopoly market structure no competitive effects can alter the repercussions of side-switching on the firm's equilibrium. Our model allows us to consider the consumers probability of side-switching as an indirect strategic variable at the firm's disposal. We will show that through an appropriate set of prices, the firm can impact the repartition of the demand into the two groups and *in fine* make additional profits. For simplicity, we assume all production costs equal to zero. Finally, the platform's budgetary constraint is given by $p_i^a + p_i^b \geq 0$, $i \in \{1, 2\}$.

2.3 Timing of actions

We consider a two-stage game. Following Armstrong (2006), we suppose that the size of each face is directly determined by a function of the consumer's utilities. The platform set two prices in the first period that maximized its profit ; one price for the face a and one price for the face b. Once the transactions are done, consumers and firm are engaged in the second period. Consumers move to the other face if they want, i.e. if their move increases their utility. In the same time, firm set the two second period prices. The two-session game is then finished. Note that pairs of profit-maximizing prices determine the relative size of each platform's face. In other words, the probability of switching from a face to the other is totally endogenous. Consumers are homo economicus ; they have no ex ante preference for being a member of one group instead of belonging to the other group. They choose the side they want to become a member by taking into account prices and externalities strength (i.e the size of the other side and the level of the externality parameters).

3 Analysis and equilibrium

3.1 First period

In the first period , the number of consumers buying the service on each side are denoted N_1^a and N_1^b and are given by the demand system :

$$\begin{cases} N_1^a := D_a(u_1^a) = D_a(\alpha_a N_1^b - p_1^a), \\ N_1^b := D_b(u_1^b) = D_b(\alpha_b N_1^a - p_1^b). \end{cases} \tag{3}$$

We assume that the functions $x \to D_a(x)$ and $x \to D_b(x)$ are increasing and differentiable on a concave support. [2] The demand on each side at monopoly prices is assumed to be positive. Then, the plarform's program is to maximize the profit of first period $\Pi_1 := p_1^a N_1^a + p_1^a N_1^b$. So the prices $(p_1^{i*},\ i = a, b)$ maximizing the profit are solutions of the system [3] :

$$p_1^{a*} = \frac{D_a}{D_a'} - \alpha_b N_1^b \text{ and } p_1^{b*} = \frac{D_b}{D_b'} - \alpha_a N_1^a. \tag{4}$$

Replacing in (3) yields the optimal demands $N_1^{a*} = D_a(\alpha_a N_1^{b*} - p_1^{a*})$ and $N_1^{b*} = D_b(\alpha_b N_1^{a*} - p_1^{b*})$. Notice that p_1^{a*} (resp. p_1^{b*}) is equal to the elasticity of the group's participation D_a/D_a' (resp. D_b/D_b'), adjusted downward by the external benefit of group b, $\alpha_b N_1^b$ (resp group a, $\alpha_a N_1^a$). The total number of consumers is then defined by $N := N_1^{a*} + N_1^{b*}$.

3.2 Second period

Our analysis of mobility is organized into two stages. At first, we shall look for equilibriums in absence of mobility, in other words, when consumers stay in their origin groups. Secondly, we shall look for equilibriums with mobility, where this mobility will be endogenous to utilities expected by consumers. Finally, we shall analyze differences between these two equilibriums and the consequences of the mobility for the firm.

2. Notice that Eq. (3) defines a unique mapping from quantities to prices but this mapping may not be invertible. For conciseness, we assume that the monopoly is able to implement any pairs of prices and of quantities that satisfy Eq. (3).

3. see Armstrong (2006).

3.2.1 Absence of mobility

In absence of mobility, the platform's program is to maximize the profit of second period $\overline{\Pi}_2 := \overline{p}_1^a N_2^a + \overline{p}_2^a N_2^b$. By definition, the consumers stay in their first period group, so we can write, $N_2^j = N_1^{j*} = D_j^*$, $j = a, b$. Where D_j^* are optimal demand functions defined in first period by equations (3) and (4).

Moreover, conditions for consumers to participate are given by

$$\overline{u}_2^a = \alpha_a D_b^* - \overline{p}_2^a \geq 0 \text{ and } \overline{u}_2^b = \alpha_b D_a^* - \overline{p}_2^b \geq 0.$$

As a consequence, the prices ($\overline{p}_2^{i*}$,$i = a, b$) maximizing the profit are given by, $\overline{p}_2^a = \alpha_a D_b$, and $\overline{p}_2^b = \alpha_b D_a$. So, the optimal profit on absence of mobility is given by

$$\overline{\Pi}_2^* = t D_a^* D_b^*. \tag{5}$$

where $t := \alpha_a + \alpha_b$ measures the totality of external effect.

3.2.2 Stochastic mobility

Let $X_1, X_2, \cdots, X_N$, N agents who have the possibility to be in face a or b. We assume that these agents make their decision independently, so that N_2^a (respectively N_2^b) the number of agents in face a (resp. b) at the second period is a sum of N independent and identically distributed random variables given by

$$N_2^a := \sum_{j=1}^{N} \mathbf{1}_{X_j = a}. \tag{6}$$

(resp. $N_2^b := N - N_2^a$), where $\mathbf{1}$ is the indicator function.

We denote $f_a := P(X_1 = a)$ and $f_b := 1 - f_a = P(X_1 = b)$, where f_j represents the proportion of consumer in face j at period 2. Otherwise, there is an equivalent interpretation where f_j represents the probability for a consumer to be in face j at the end of the second period. Obviously the mean of N_2^a denoted $\mathrm{E}(N_2^a)$ is given by $\mathrm{E}(N_2^a) = f_a N$, and similarly $\mathrm{E}(N_2^b) = N - f_a N = f_b N$. However, the mobility is a dynamic process, and we assume that there is a price couple which ensures a stable equilibrium at the end of the second period.

Definition 3.1 ***Equilibrium at second period***
The prices (p_2^a, p_2^b) define the second peridod equilibrium prices if
i) No agent can improve his utility by changing face, and
ii) The mean profit of the platform is maximal.

Condition i) guarantees an optimal equilibrium for the consumers, and condition ii) ensures an optimal equilibrium for the platform. So, when i) and ii) are satisfied, the mobility stops, because neither the platform, nor consumers can hope for a better gain. It is what we define as a stable equilibrium. Definition 3.1 implies the following

Proposition 3.1 *If prices (p_2^a, p_2^b) assure equilibrium at second period, then, $u_2^a(p_2^a, p_2^b) = u_2^b(p_2^a, p_2^b)$.*

Proof
Let (p_2^a, p_2^b) the prices, defined in Definition 3.1, assuring the equilibrium at second period.
If $u_2^a > u_2^b$, then agents migrate in face a. So, N_2^a increases and N_2^b decreases. Therefore by the external effects and as the prices are fixed, u_2^a decreases and u_2^b increases. According to Definition

3.1, this process is stopped when $u_2^a = u_2^b$.
In a similar way we prove that $u_2^a < u_2^b$ is impossible.
□

Proposition 3.1 implies that at equilibrium both faces bring the same utility to consumers. Indeed, for a given set of prices, external effects make converge the levels of utilities. We are now able to obtain explicit expressions for the proportions $(f_j, j = a, b)$.

Corollary 3.1
Let the prices (p_2^a, p_2^b) *assure one equilibrium at second period. We have,*

$$f_a = \frac{p_2^b - p_2^a}{Nt} + \frac{\alpha_a}{t},$$

$$f_b = \frac{p_2^a - p_2^b}{Nt} + \frac{\alpha_b}{t},$$

with the conditions $0 \leq f_a \leq 1$ *and* $0 \leq f_b \leq 1$.

Proof : See appendix.

We can see that $(f_j, j = a, b)$ are endogenous to prices and external effects. Also we notice that the proportion of agents migrating in face j increases with the utility of group j. For example, when the price of the group a (p_2^a) increases, ceteris paribus, f_a decreases and f_b increases as $f_b = 1 - f_a$. The intuition for this is that when p_2^a increases, ceteris paribus, the expected utility for the group a decreases, therefore, consumers are less incited to move to group a, thus, f_a decreases. The condition on the Corollary 3.1 easily implies

Lemma 3.1

$$\left.\begin{array}{l} 0 \leq f_a \leq 1 \\ 0 \leq f_b \leq 1 \end{array}\right\} \Leftrightarrow -N\alpha_b \leq p_2^a - p_2^b \leq N\alpha_a.$$

Proof : Obvious according to the Corollary 3.1.

The computations of endogenous probabilities show that consumers go to the other face considering price's faces differential and the relative strength of network externalities. Not surprisingly, we see that a consumer is more likely to move to the cheapest face and also to the group that confers the highest externality.

In our model, the platform's profit at the second period is a random variable given by $\tilde{\Pi}_2 = p_2^a N_2^a + p_2^b N_2^b$. Moreover the platform's behavior is to maximize the mean of Π_2 given by

$$\Pi_2(p_2^a, p_2^b) := \mathrm{E}[\tilde{\Pi}_2(p_2^a, p_2^b)] = p_2^a N f_a + p_2^b N f_b,$$

where $\mathrm{E}(\cdot)$ is the expectation. That is to say, according to Corollary 3.1,

$$\Pi_2(p_2^a, p_2^b) = \frac{N(p_2^a\alpha_a + p_2^b\alpha_b)}{t} - \frac{(p_2^a - p_2^b)^2}{t}. \tag{7}$$

We now resume the different constraints for the second period, the first come from Lemma 3.1. Then there is consumers' participation constraints given by

$$\left.\begin{array}{l} \mathrm{E}(u_2^a) \geq 0 \\ \mathrm{E}(u_2^b) \geq 0 \end{array}\right\} \Leftrightarrow p_2^a\alpha_b + p_2^b\alpha_a \leq \alpha_a\alpha_b N.$$

Finally the constraint of the platform's budget is given by $p_a^2 + p_b^2 \geq 0$. Therefore, the space of prices at second period is given by

$$C_2 := \{(p_2^a, p_2^b) \in \mathbb{R}^2 | p_2^a \alpha_b + p_2^b \alpha_a \leq \alpha_a \alpha_b N; -N\alpha_b \leq p_2^a - p_2^b \leq N\alpha_a; p_2^a + p_2^b \geq 0\}. \quad (8)$$

As a consequence, the optimal prices (p_a^{2*},p_b^{2*}) at equilibrium should satisfy

$$\{p_a^{2*}, p_b^{2*}\} := \underset{(x,y)\in C_2}{\operatorname{argmax}} \Pi_2(x, y).$$

Theoreme 3.1 *(prices at equilibrium) We have*

$$p_2^{a*} = \alpha_a \frac{N}{2} \quad \textit{and} \quad p_2^{b*} = \alpha_b \frac{N}{2}.$$

As a consequence the optimal proportion are given by $f_a^* = f_b^* = \frac{1}{2}$, *and the mean of the profit at equilibrium by*

$$\Pi_2^* := E(\tilde{\Pi}_2^*) = \frac{N^2 t}{4}. \quad (9)$$

Proof : See appendix.

$\tilde{\Pi}_2^*$ is a random walk with increments p_2^{b*} or p_2^{a*} with probability $1/2$. In particular we easily obtain the variance of $\tilde{\Pi}_2^*$ [4]

$$\mathrm{Var}(\tilde{\Pi}_2^*) = \frac{N}{4}(p_2^{a*} - p_2^{b*})^2 = \frac{N^3}{16}(\alpha_a - \alpha_b)^2.$$

It is interesting to remark that mobility introduces uncertainty both for firm and consumers. The mobility gives birth to a specific kind of risk : the possibility that the group which gives utility (or profit) has a decreasing size in the future. For the monopoly, the variance of expected profit is increasing with the differential of externality parameters (α_a and α_b). The larger is the difference between α_a and α_b the larger is the gap between expected prices p_a^{2*} and p_b^{2*}. For this reason, the firm could realize a smaller turnover if consumers go massively to the cheapest face. For consumers, the expected utility depends on the size of the other group. The higher is the consumers' mobility the higher is the variance of the size of the groups.

Thanks to Theorem 3.1 we can easily determine the impact of mobility on the platform's income. We call S the extra profit that the firm can earn from the consumer's mobility :

Corollary 3.2

$$S := E(\tilde{\Pi}_2^* - \overline{\Pi}_2^*) = \frac{t}{4}(N_1^{a*} - N_1^{b*})^2$$

Proof : See appendix.

4. As the probability f_a^* optimizing the problem is $1/2$, This implies that the distribution of N_2^a (which maximizes the mean of the profit) is a Binomial with parameters N and $1/2$. and as the profit can be written

$$\tilde{\Pi}_2 = p_2^b N + (p_2^a - p_2^b) N_2^a = \sum_{j=1}^{N} (p_2^b + (p_2^a - p_2^b) 1_{X_j = a}).$$

which is a simple transformation of N_2^a.

There are two noteworthy implications of this extra profit. First, we notice that the larger is the spread between the initial sizes of groups, the greater is the profit. Remember that mobility is determined by the expected utility of consumers. For example, suppose that $N_1^a > N_1^b$. Then, a consumer belonging to the group a can improve his satisfaction by moving into group b in order to obtain a stronger network effect. On the contrary, a consumer belonging to the group b can improve his utility by moving into group a so as to take benefit from lower price. Therefore, the larger is the spread between initial sizes of groups, the higher are the consumers' incentives to move from one group to the other one. In other words, the larger is the gap between the two populations of consumers in period one, the larger is the expected utility of the second period. The consumers' reservation price is then increasing with the difference in size of group-a and group-b. In view of non-competitive market structure, the firm is able to capture the augmentation of willingness to pay. The firm finally earns a mobility rent.

Second and not surprisingly implication, we notice that the profit is increasing with the sum of external effects t, with $t := \alpha_a + \alpha_b$.

4 Discussion : Mobility as a practical management strategy ?

If mobility is not possible, standard TSM economics show that firm adjusts prices to the level of externality that a group of consumers generates. The central idea is that the price is adjusted by the external benefit that a group brings to the other group (the more a consumer brings value to the other group the smaller is the price). We can see this specific way of setting prices as "a reward pricing". If mobility is possible, modification in size group is a mechanism that raises profit in itself. The firm receives incitation to set prices that homogenizes the number of consumers in each group. At the end of this process (that is to say in period 2), there is no "reward pricing" anymore. Externalities parameters reflects then consumers' disposition to pay and are transforming into profits. Can we find in the industry some illustrations of these special relationships between the firm and their consumers ? Do the firms really encourage mobility ?

The booking site Airbnb is a good example of a TSM with mobility. Airbnb's simplicity has made it easy for people to find accommodation in private residences and to book them conveniently. Consumers can discover the site during a trip by the renter side and then, once back home, become an offeror of a room in his flat, i.e. make a side-switching. The same mechanism prevails for the electronic payment system Paypal. Paypal users can receive or sent funds (i.e. make a side-switching) very easily, with very low transaction costs and almost no constraints. That is true as well for the emblematic electronic platform Ebay which easily permits its consumers to become sellers through the Amazon Marketplace program. We think that these very important common features of Ebay, PayPal and Airbnb (and other major electronic platforms) imply similarity in business management for these successful firms. Mobility is quality ; it is one of the reasons why these firms develop advertising programs, discount prices or didactic tutorials to give incentives to consumers to make some business into the other side of their favorite platform. We think that our model is consistent with these observations. Consumers' side-switching is a natural way to develop the business but it is also an attractive method of consumers' surplus extraction.

5 Conclusion

The possibility of changing from a face to the other appears to consumers like an improvement of product's quality. Since the platform is a monopoly, it can capture the increase of the consumer's utility and make additional profit compared to the first period of time. In this model, mobility is a valuable feature for consumers which enhance quality's product. Actually, this is a strong economic explanation for the profitability of the mobility. Finally our paper shows that, in a specific context, mobility is a strategical variable in the TSM industry.

According to our results, it would be interesting to study the TSM in a dynamic model. For example, we would define an intertemporal "cyclic" model with two phases. During phase one, we can imagine a firm that encourage the dissymmetry between the two sides. For example, the platform could attract many consumers to one side in the case of a new market. In the case of an existing market, the platform would encourage the whole population to move (or stay) to a certain side of its business. Then in the second phase, and according to our model's results, we could imagine a firm which takes advantage of the initial gap of the two groups. With an adequate pricing, the platform would equalize the groups' average sizes and then start a "convergence" phase. Dynamic modeling could determine conditions under which an alternating of dissymmetry and convergence phases should be a profitable strategy for the firm.

6 Appendix

Proof of Corollary 3.1

According to Proposition 3.1 $u_2^a(p_2^a, p_2^b) = u_2^b(p_2^a, p_2^b)$, so, $\mathrm{E}(u_2^a(p_2^a, p_2^b)) = \mathrm{E}(u_2^b(p_2^a, p_2^b))$ where $\mathrm{E}(\cdot)$ is the expectation. Recalling (1) we obtain $\mathrm{E}[u_2^a(p_2^a, p_2^b)] = \alpha_a E N_2^b - p_2^a = \alpha_a N f_b - p_2^a$, and, $\mathrm{E}[u_2^b(p_2^a, p_2^b)] = \alpha_b N f_a - p_2^b$. As $f_a + f_b = 1$, we obtain

$$\begin{cases} \alpha_a N f_b - p_2^a = \alpha_b N f_a - p_2^b, \\ f_a + f_b = 1. \end{cases} \Leftrightarrow \begin{cases} f_a = \frac{p_2^b - p_2^a}{Nt} + \frac{\alpha_a}{t}, \\ f_b = \frac{p_2^a - p_2^b}{Nt} + \frac{\alpha_b}{t}. \end{cases}$$

$\square$

Proof of Proposition 3.1

The platform's program at the second period is given by the minimization problem (P) :

$$\min_{(x,y)\in C_2} -\Pi_2(x,y) \quad (P),$$

where the mean profit Π_2 is the continuously differentiable function given by (7) and C_2 is given by (8). First note that C_2 is a compact space of $\mathbb{R}^2$, therefore (P) admits at least one solution. Also the function Π_2 does not admit critical points : it does not exist $(x,y) \in C_2$ such that $\nabla\Pi_2(x,y) = 0$, where ∇ is the gradient. Consequently, the solution of the problem (P) is on the boundary of the space C_2, that we denote ∂C_2. The Lagrangien L of (P) is given by

$$\begin{aligned} L(x,y,\lambda) = &- \Pi_2(x,y) + \lambda_1(p_2^a\alpha_b + p_2^b\alpha_a - \alpha_a\alpha_b N) - \lambda_2(p_2^a + p_2^b) \\ &+ \lambda_3(-N\alpha_b - p_2^a + p_2^b) + \lambda_4(p_2^a - p_2^b - N\alpha_a), \end{aligned}$$

where $\lambda = \{\lambda_i; i = 1 \cdots 4\}$ represent lagrange's multipliers. Then, Karush-Kuhn-Tucker (KKT) first order conditions are given by the following system,

$$\begin{cases} \partial_x L = -\frac{N}{t}\alpha_a + \frac{2}{t}(p_2^a - p_2^b) + \alpha_b\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 = 0, \\ \partial_y L = -\frac{N}{t}\alpha_b - \frac{2}{t}(p_2^a - p_2^b) + \alpha_a\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 = 0, \\ \lambda_1(p_2^a\alpha_b + p_2^b\alpha_a - \alpha_a\alpha_b N) = 0, \\ \lambda_2(p_2^a + p_2^b) = 0, \\ \lambda_3(-N\alpha_b - p_2^a + p_2^b) = 0, \\ \lambda_4(p_2^a - p_2^b - N\alpha_a) = 0. \end{cases} \tag{10}$$

The existence of one TSM implies that $\lambda_3 = \lambda_4 = 0$. Indeed, if it is not the case, the totality of the consumers may move to the same face, and the TSM disappear. So we now deal with the case $\lambda_2 = 0$ and then $\lambda_1 = 0$. First, if $\lambda_2 = 0$ and $\lambda_1 > 0$ the system (10) becomes,

$$\begin{cases} \partial_x L = -\frac{N}{t}\alpha_a + \frac{2}{t}(p_2^a - p_2^b) + \alpha_b\lambda_1 = 0, \\ \partial_y L = -\frac{N}{t}\alpha_b - \frac{2}{t}(p_2^a - p_2^b) + \alpha_a\lambda_1 = 0, \\ p_2^a\alpha_b + p_2^b\alpha_a = \alpha_a\alpha_b N. \end{cases}$$

After resolution of this system, we find

$$\begin{cases} p_2^a = \frac{N}{t}\alpha_a, \\ p_2^b = \frac{N}{t}\alpha_b, \\ \lambda_1 = \frac{N}{t} > 0. \end{cases}$$

If $\lambda_2 > 0$ and $\lambda_1 = 0$, the system (10) becomes,

$$\begin{cases} \partial_x L = -\frac{N}{t}\alpha_a + \frac{2}{t}(p_2^a - p_2^b) - \lambda_2 = 0, \\ \partial_y L = -\frac{N}{t}\alpha_b - \frac{2}{t}(p_2^a - p_2^b) - \lambda_2 = 0, \\ p_2^a = -p_2^b. \end{cases}$$

Once the system is solved, we find $\lambda_2 = -\frac{N}{2} < 0$ which is impossible because Lagrange multipliers are positive. So in this case, there are no points who verify KKT conditions. Finally, there is only one minimum $p_0 := (N/t \cdot \alpha_a, N/t \cdot \alpha_b)$ complying the first order conditions and the constraint of TSM's existence. Taking $(p_2^a, p_2^b) = p_0$, we have,

$$f_a = \frac{p_2^b - p_2^a}{Nt} - \frac{\alpha_a}{t} = \frac{1}{2} = f_b,$$

and

$$\Pi_2(p_0) = p_2^a f_a N + p_2^b f_b N = \frac{N^2 t}{4}.$$

Finally, it is obvious that $\mathcal{O} := (0,0) \in C_2$, and $\Pi_2(\mathcal{O}) = 0$. So, there is a point $(\mathcal{O})$, such as $\Pi_2(\mathcal{O}) > -\Pi_2(p_0) = -N^2 t/4$, so, p_0 is a local minima of $-\Pi_2$, so a local maxima of Π_2. As a consequence, we have $\Pi_2^* = \Pi_2(p_0)$.

□

Proof of Corollary 3.2
According to (5) and (9), we have

$$\mathrm{E}(\tilde{\Pi}_2^* - \overline{\Pi}_2^*) = \frac{tN^2}{4} - tN_1^{a*}N_1^{b*}. \tag{11}$$

Also according to (2), $N := D_a^* + D_b^*$. As a consequence, (11) becomes

$$\mathrm{E}(\tilde{\Pi}_2^* - \overline{\Pi}_2^*) = \frac{t}{4}(N_1^{a*} - N_1^{b*})^2.$$

□

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