When more does not necessarily mean better: Health-related illfare comparisons with non-monotone welfare relationships

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Abstract

Most welfare studies are based on the assumption that wellbeing is monotonically related to the variables used for the analysis. While this assumption can be regarded as reasonable for many dimensions of wellbeing like income, education, or empowerment, there are some cases where it is definitively not relevant, in particular with respect to health. For instance, health status is often proxied using the Body Mass Index (BMI). Low BMI values can capture undernutrition or the incidence of severe illness, yet a high BMI is neither desirable as it indicates obesity. Illfare estimattions derived from usual poverty measures are then not appropriate. This paper proposes poverty indices that are consistent with some situations of non-monotonic wellbeing relationships and examines the partial orderings of different distributions derived from various classes of poverty indices. An illustration is provided for health-related illfare as proxied by the BMI using DHS data for Bangladesh during the period 1997–2007. It is shown that the gains of the decline of undernutrition are undermined by the rapid increase of obesity.

Keywords: Illfare comparisons, poverty measurement, stochastic dominance, monotonicity, Bangladesh.
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1 Introduction

Target 1.C from the Millennium Development Goals states that the proportion of people who suffer from hunger should be halved between 1990 and 2015. Although this objective is presumed not to be met in 2015, the share of undernourished individuals has declined during the period(de Onis et al., 2004, Dpt of Ec. and Soc. Affairs of the U. N. Secretariat, 2012). For instance, the FAO finds that the share of undernourished people in the developing world fell from about 20% to 15% during the period 1990-2010.[1] However, a stylized fact in most developing countries is that progresses with respect to undernutrition have often be associated with increase in obesity (Popkin et al., 2012). This so-called nutrition transition process raises the issue of a net gain in social welfare with respect to health issues. Should we consider that the level of welfare in a society has improved if undernutrition has declined but other forms of malnutrition have become more severe? If we want to get a global assessment of the social progresses of a given population with respect to nutrition, we then need to be able to make socially comparable the situations of underweighted and overweighted individuals.

Wellbeing is generally supposed to be monotonically related to the variables used for the analysis in poverty and welfare studies. While this assumption can be regarded as reasonable for many dimensions of wellbeing like income, education, or empowerment, there are some cases where it is definitively not relevant, in particular with respect to health. For instance, health status is often proxied using the Body Mass Index (BMI) in the case of adults,[2] or using weight-for-age or height-for-age in the case of children and adolescents. Low BMI values can capture undernutrition or the incidence of severe illness, yet a high BMI is neither desirable as it indicates obesity. That is why the BMI is usually compared against a left-tail and a right-tail cut-off which work as deprivation lines, e.g. 18.5 kg/m^2 and 25 kg/m^2, respectively. Estimating aggregate illfare using traditional poverty indices, based on a unique (left-tailed) deprivation line, are then not appropriate. Likewise several other health indicators are characterized by the use of two deprivation lines for diagnostic purposes because they relate to situations in which either "having too much" or "too little" is detrimental to health. That is the case of several blood tests, including blood pressure, Thyroid function, hemoglobin and total cholesterol.[3]

This paper first proposes illfare indices that are consistent with situations of non-monotonic relationships between wellbeing and its indicators, i.e. like the measures of wellbeing mentioned above. These indices are decomposable into two indices that, respectively, measure a concept of "loss" illfare and another one of "excess" illfare. While "loss" illfare is identical to the traditional understanding of poverty as insufficiency, "excess" illfare refers to wellbeing harmed by suboptimal abundance. The family of indices is ax-

[1] Figures are from the 2012 Millenium Development Goals Report (Dpt of Ec. and Soc. Affairs of the U. N. Secretariat, 2012).

[2] The BMI, also known as the Quetelet index, is defined as the individual's body mass (in kilograms) divided by the square of his or her height (in meters).

[3] It was suggested during a seminar at the CERDI that our framework could also be applied to free time. Both having too much free time or not having it enough could be source of major stress and result in low wellbeing levels. On time poverty, see for instance XXXXXX.

iomatically characterized and includes extensions to traditional poverty indices like the Foster-Greer-Thorbecke family and the Watts index. For the purpose of characterization we introduce key alterations to the traditional axioms of focus, monotonicity and transfers.

Indices provide precise and useful informations as well as a complete ordering of observed distributions. However, they all are based on specific underlying welfare functions (Blackorby and Donaldson, 1980) upon which agreement may not be met. In the health context, risks of death or severe disease may of course be precisely estimated for the the different values of the variable under consideration, but it is not so clear how people value such risks in terms of wellbeing. The relationship becomes even more complex once psychological and social aspects of health are taken into account. For these reasons, it is necessary to look for criteria that make it possible to draw robust conclusions about the state of illfare, that is to obtain results that do not depend on the specific functional forms used to assess illfare. The paper also examines the partial orderings of different distributions, according to sub-families of our class of illfare indices, by deriving the required first and second-order stochastic dominance conditions. We also study the conditions for partial orderings when the experience of one form of illfare (e.g. "loss" illfare) is considered to be worse than the other one (e.g. "excess" illfare).

The rest of the paper is organized as follows: The next section introduces the family of non-monotone illfare indices and its associated partial ordering conditions. The third section proposes stochastic dominance conditions when the two forms of illfare are deemed to have differential effects on wellbeing. Section 4 shows how to compute the standard errors for the family of indices and the fifth section provides an empirical illustration using Bangladeshi data from the Demographic Health Survey (DHS) for the period 1997–2007. It is shown that health-related illfare levels have declined during the period for both mothers and under-5 children but that the overall alleviation is partly offset by the increase of obesity. The paper concludes with some final remarks.

2 Non-monotone poverty measurement: The general case

2.1 Two classes of poverty indices with revised versions of the focus, monotonicity and transfer axioms

Illfare is usually used as a synonym for poverty with poverty being defined as a lack of resources with respect to a socially defined norm. Let's x describe an individual attribute defined on the domain $\Omega := [\omega^-, \omega^+] \subset \Re$. Illfare may then be assessed using unidimensional additive poverty indices $P(z)$ that are of the type:

$$P(z) := \int_{\omega^-}^{z} \pi(x,z)\, dF(x), \tag{1}$$

where F is the cumulative distribution function (cdf), $z \in \Omega$ is the poverty line, and π :

$\Omega \times \Omega \rightarrow \Re_{+}$ is an individual poverty index such that:

$$\pi(x,z) \begin{cases} \geqslant 0 & \text{if } x \leqslant z, \\ = 0 & \text{otherwise.} \end{cases} \tag{2}$$

Indices (1) satisfy the traditional properties of continuity, anonymity, population replication, focus and decomposability. Moreover, they also comply with weak monotonicity if $\frac{\partial \pi}{\partial x} \leqslant 0$. In general the monotonicity axiom meets a large agreement and is consistent with poverty assessments based on income.

With indices (1), illfare is associated with insufficient level of the variable x with regard to a norm corresponding to z. However, the relevant space for conceptualizing wellbeing is rarely the one where attribute x is defined. Indeed, the "failure to achieve certain minimum capabilities" (Sen, 1985) does not systematically mean an insufficient value for x. So, in the space of capabilities, illfare can be defined as a lack of resources but potentially not in the space of x. Considering nutrition, a person is health-deprived if she does not have the ability to get an adequate and balanced diet, regarding her physiological, psychological and social needs. Causes of this inability are diverse, including for instance low income, limited access to diversified sources of nutrients, low information with respect to the importance of a balanced diet, severe diseases or handicaps, and psychological disorders. Whatever, the exact roots for health-related illfare, we consider that they are the expression of low capabilities.[4]

Here we consider illfare indices that do not exhibit the same behaviour as indices (1) because the underlying relationship between variable x and welfare is not supposed to be monotonic. More specifically, we introduce a set of deprivation lines $\{z^L, z^U\} \subset \Omega$, with $z^L < z^U$, such that:[5]

$$\pi(x; z^L, z^U) \begin{cases} \geqslant 0 & \text{if } x \leqslant z^L, \\ = 0 & \text{if } x \in (z^L, z^U), \text{ and} \\ \geqslant 0 & \text{if } x \geqslant z^U. \end{cases} \tag{3}$$

[4] Low capabilities can be a valuable definition for poverty, but in the present study we prefer using the term "illfare". Indeed, although overweight is often associated with low income in developed countries (see for instance **?**), being fat in many low-income countries is traditionally regarded as a sign of economic success (see for instance XXXX). Such situations can be seen as an illustration of the discrepancy between the private and social evaluations of life. Tough people may value more the exhibition of richness signs that the potential risks of overweight, it is reasonable to consider that the social evaluator should only take the health aspects into account. However, recent evidence shows that, even in developing countries, obesity tends to become more related to monetary poverty (Popkin et al., 2012). Moreover, clinical evidence shows that undernutrition of mother during pregnancy as well as undernutrition during childhood result in a higher probability of obesity for the children.

[5] Here we suppose that the same deprivation lines z^L and z^U can be applied for each individual within the observed populations and are exogenous with respect to the observed values of x within these populations. The first assumption means that the same thresholds can be applied for each person whatever her sex, age, or any other relevant characteristic. Both for poverty measurement and dominance tests, that assumption can be slackened, notably by rescaling observed values of x so that all group-specific poverty lines coincide.

The second assumption signifies that we are only considering absolute poverty measurement. While this hypothesis is very reasonable considering the physiological dimension of health, we must acknowledge that is debatable once the psychological and social aspects are taken into account. For instance, we can admit that being fat is socially a more acute issue when obesity is rare than when it is widely spread among the population. Those considerations are however left aside for further work.

Illfare thus relates in the present paper to situations in which either “having too much” or “too little” is detrimental for a person’s wellbeing. We note at the outset that such non-monotone relationship with respect to health has already been investigated regarding health-inequalities (Dutta, 2007, see for instance), but, at the best of our knowledge, no tool has been proposed for the social assessment of the burden of poor health-related situations.

At the social aggregation level, we consider illfare indices P of the type:

$$P(z^L, z^U) := \int_{\omega^-}^{z^L} \pi(x; z^L, z^U)\, dF(x) + \int_{z^U}^{\omega^+} \pi(x; z^L, z^U)\, dF(x). \tag{4}$$

It first can be observed that the definition of P proposed in equation 1 can be seen as the limiting case $z^U = \omega^+$ of the definition presented in equation 4. It is also worth noting that P in equation (4) does not fulfil the traditional definitions of the focus and monotonicity axioms proposed in the seminal article by Sen (1976) for poverty measurement. A poverty index is said to comply with the focus axiom if the poverty level does not change after an x-increment for a non-poor person. However for any individual with $x \in [z^L, z^U]$, there is always an increment $\kappa > 0$ such that $x + \kappa \geqslant z^U$, *i.e.* the individual falls into illfare. In the same manner, the monotonicity axiom is usually defined in such a way that the increase of the value of the wellbeing attribute x for a poor person does not increase poverty. Nevertheless with our setting we suppose that increases above the upper poverty line z^U should not decrease poverty. These conflicts are not surprising as the focus and monotonicity axioms are usually defined for indices like those of the form of equation (1). Since the focus and monotonicity axioms express simple and desirable properties, it is worth proposing new definitions for these axioms so as to fit our specific framework. Formally:

Axiom (FOC). *$P_A(z^L, z^U) = P_B(z^L, z^U)$ if distribution B is obtained from distribution A by adding $\kappa \in \Re$ to any observed value $x \in (z^L, z^U)$ such that $x + \kappa \in (z^L, z^U)$.*

Axiom (MON). *$P_A(z^L, z^U) \leqslant P_B(z^L, z^U)$ if distribution B is obtained from distribution A i) by substracting $\kappa > 0$ to any observed value $x \in [\omega^-, z^L]$ such that $x - \kappa \in \Omega$, or ii) by adding $\kappa > 0$ to any observed value $x \in [z^U, \omega^+]$ such that $x + \kappa \in \Omega$.*

Axioms FOC and MON are then defined in order to preserve the spirit underlying their usual definitions. With FOC it is assumed that a change in x for a non-poor person does not change poverty as long as the person remains non-poor. The monotonicity axiom is usually defined to state that movements towards the poverty line for a poor person do not increase poverty. That is exactly what axiom MON says. To explain that point, let us introduce the concepts of "loss" poverty and "excess" poverty. The former refers to an insufficient amount of a wellbeing attribute x, usually judged by comparing against the left-tail poverty line z^L. By contrast, "excess" poverty is the situation of an excessive, and detrimental, amount of a wellbeing attribute, or indicator, e.g. health indicators like BMI. It is determined by comparing x against the right-tail poverty line z^U. Then our monotonicity axiom states that both a decrease in x for a "loss" poor person, and an increase in x for an "excess" poor person do not decrease overall poverty.

We can now define the following class of non-monotone illfare indices:

$$\Pi^1(z^{L+}, z^{U-}) := \left\{ P \left| \begin{array}{l} [z^{L+}, z^{U-}] \subseteq [z^L, z^U] \subset \Omega \\ \pi(z; z^L, z^U) = 0 \forall z \in \{z^L, z^U\} \\ \pi^{(1)}(x; z^L, z^U) \leqslant 0, \ \forall x \leqslant z^{L+}, \text{ and } \pi^{(1)}(x; z^L, z^U) \geqslant 0, \ \forall x \geqslant z^{U-} \end{array} \right. \right\}, \quad (5)$$

where $\pi^{(1)}(x; z^L, z^U) \equiv \frac{\partial \pi}{\partial x}$. Members from $\Pi^1(z^{L+}, z^{U-})$ fulfills FOC and MON as defined in the present paper. They also comply with the traditional anonymity, additivity, continuity and population invariance axioms. Anonymity states that x is the sole characteristics that can explain why two individuals could exhibit differing values of π. Thus, other characteristics like age, household size, ethnolinguistic distinctive features or gender should not be taken into account when assessing poverty. Additivity means that overall poverty is the arithmetic mean of poverty estimates at the individual level, a property that is desirable within our framework in order to assess the relative contribution of "loss" and "excess" poverty to overall poverty. Continuity at the poverty is the result of the second condition in (5) and is necessary to avoid small measurement errors to produce non-marginal variations in the estimated poverty level.[6] Finally, the population invariance principle states that replicating each member of the population the same number of times does not change the level of poverty, so that population of different size can be compared in terms of poverty.

Interesting examples of $P \in \Pi_1(z^{L+}, z^{U-})$ are the following extensions of the traditional Watts's (1968) and Foster et al.'s (1984) poverty indices:

$$W_\beta(z^L, z^U) := \int_0^{z^L} \log \frac{z^L - \omega^-}{x - \omega^-} dF(x) + \beta \int_{z^U}^{\omega^+} \log \frac{\omega^+ - z^U}{\omega^+ - x} dF(x), \quad (6)$$

$$FGT_{\beta,\alpha_L,\alpha_U}(z^L, z^U) := \int_{\omega^-}^{z^L} \left(\frac{z^L - x}{z^L - \omega^-} \right)^{\alpha_L} dF(x) + \beta \int_{z^U}^{\omega^+} \left(\frac{x - z^U}{\omega^+ - z^U} \right)^{\alpha_U} dF(x), \quad (7)$$

with $\beta > 0$, $\alpha_L \geqslant 1$, and $\alpha_W \geqslant 1$. The family $FGT_{\alpha_L,\alpha_U}(z^L, z^U)$ also includes the headcount index for $\alpha_L = \alpha_W = 0$, that is not member from $\Pi_1(z^{L+}, z^{U-})$ as it is not continuous within the poverty domain but provides useful information about the spread of poverty within the population. β is a weighing parameter that gives more emphasis on "loss" poverty for $\beta \in (0, 1)$ and on "excess" poverty for $\beta > 1$. The parameters α_L and α_U relates to the sensitivity to extreme forms of deprivations.

These indices are relative indices as the size of individual deprivations is normalized by the corresponding value for the maximum deprivation. In the case such a normalization procedure is not regarded as desirable, one may prefer for instance the following absolute version of the $FGT_{\alpha_L,\alpha_U}(z^L, z^U)$:

$$FGT^A_{\beta,\alpha_L,\alpha_U}(z^L, z^U) := \int_{\omega^-}^{z^L} \left(z^L - x\right)^{\alpha_L} dF(x) + \beta \int_{z^U}^{\omega^+} \left(x - z^U\right)^{\alpha_U} dF(x), \quad (8)$$

[6] We can note at the outset that continuity at the poverty line is not necessary for the design of first order stochastic conditions. Consequently, the conditions expressed in Proposition 1 could also be applied to a broader class of poverty indices that may not respect continuity at the poverty line. On the other hand, continuity is desirable for second order dominance conditions. On this specific point, see for instance Araar and Duclos (2006).

with $\alpha_L \geqslant 0$ and $\alpha_U \geqslant 0$.

In line with Sen (1976) we may prefer poverty indices to be sensitive to inequalities between the poor. Such distribution-sensitive indices are then supposed to comply with a transfer axiom that states that progressive transfers between two-poor individuals should decrease, or at least not increase, poverty. However, it is worth noting that, contrary to poverty indices of the type (1), Pigou-Dalton transfers have to be considered over a non-convex set within our framework since the poverty domain is defined by the union of distinct intervals. Consequently, we may consider three cases: *i)* when both people are "loss" poor; *ii)* when both are "excess" poor; and *iii)* when the two poor belong to different groups. The first two cases are not a matter of concern since progressive transfers may not result in exits from poverty and it is always possible to rank the two people in terms of poverty. However, in the third case, one cannot say *a priori* whether the "loss" poor is worst off when compared with the "excess" poor. Consequently, a transfer from the former to the latter may either be deemed progressive or regressive and it is not possible to consider that case when defining the transfer axiom within our framework. Fortunately, this situation is not worth considering since a transfer from the "excess" poor to the "loss" poor means wellbeing improvements for both people and thus can be addressed using MON. Hence the inability of our transfer axiom to deal with transfers between any pair of poor individuals is not a a matter of concern since we always consider poverty indices that comply with MON in the present paper.

The transfer axiom can thus be presented in the following manner:

Axiom (TRA). *$P_A(z^L, z^U) \geqslant P_B(z^L, z^U)$ if distribution B is obtained from distribution A by transferring $\kappa > 0$ from individual i to individual j such that $\{x_i, x_j\} \subset [\omega^-, z^L]$ or $\{x_i, x_j\} \subset [z^U, \omega^+]$, and $|x_i - x_j| \geqslant |(x_i - \kappa) - (x_j + \kappa)|$.*

It is worth indicating that the index $W(z^L, z^U)$ complies with TRA while members from the class of indices $FGT_{\alpha_L,\alpha_U}(z^L, z^U)$ respect this transfer axiom only for $\alpha_L \geqslant 2$ and $\alpha_U \geqslant 2$.

If we want poverty not to increase in the aftermath of Pigou-Dalton transfers and thus to fulfill TRA, then we can consider the following class:

$$\Pi^2\left(z^{L+}, z^{U-}\right) := \left\{ P \in \Pi^1(z^{L+}, z^{U-}) \;\middle|\; \begin{array}{l} \pi^{(1)}\left(z; z^L, z^U\right) = 0 \forall z \in \{z^L, z^U\} \\ \pi^{(2)}\left(x; z^L, z^U\right) \geqslant 0, \ \forall x \in \Omega \end{array} \right\}, \tag{9}$$

where $\pi^{(2)}\left(x; z^L, z^U\right) \equiv \frac{\partial^2 \pi}{(\partial x)^2}$. The first condition is basically a continuity assumption. The second condition in (9) captures the requirement regarding the sensitivity of the social poverty function to progressive transfers. In formal terms, the additivity of P associated with the second condition in (9) means that members from $\Pi^2\left(z^{L+}, z^{U-}\right)$ are S-convex in "loss" poverty values of x and also S-convex in "excess" poverty values of x. Both conditions mean finally that the marginal gain in the improvement of the situation of a poor person decreases and tends to zero as she moves closer to its deprivation line. It can regarded as a desirable property for a poverty index as it fosters policy efforts on those individuals experiencing severe "losses" or "excesses."

2.2 Partial orderings

The very limited set of conditions expressed for the definition of the classes $\Pi^1\left(z^{L+}, z^{U-}\right)$ and $\Pi^2\left(z^{L+}, z^{U-}\right)$ leaves the door open for a great variety of poverty indices, and modified Watts and FGT indices are only suggestions of what could be an appropriate index to assess the extent of poverty within our non-monotone framework. In the following paragraphs, we consider stochastic dominance conditions that make it possible to get robust results when performing ordinal poverty comparisons, that is results that do not hinge on specific poverty indices or poverty lines choices. We first propose a set of criteria for the class of poverty measures Π^1.

Proposition 1.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \forall P \in \Pi^1(z^{L+}, z^{U-}) \tag{10}$$

$$\textit{iff} \qquad F^A(x) \leqslant F^B(x) \quad \forall x \in [\omega^-, z^{L+}] \tag{11}$$

$$\textit{and} \qquad F^A(x) \geqslant F^B(x) \quad \forall x \in [z^{U-}, \omega^+]. \tag{12}$$

Proof. See appendix A.1 ■

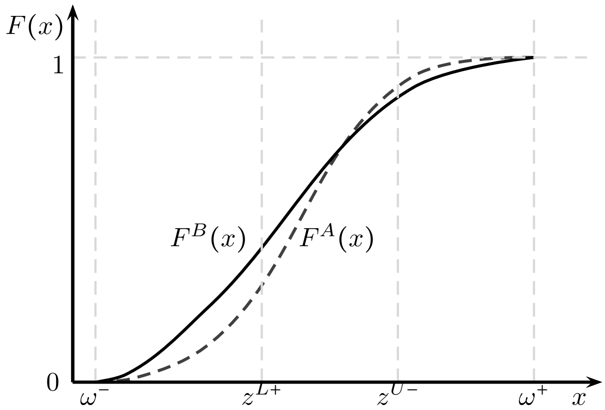

Figure 1: First order dominance

The first-order dominance relationship presented in Proposition 1 states that poverty in distribution A is not higher than in distribution B if the value of the "loss" poverty headcount index is never larger for distribution A for each value of the poverty line within the largest admissible "loss" poverty domain $[\omega^-, z^{L+}]$ and if the "excess" poverty headcount is never higher in A for each poverty line within the largest admissible "excess" poverty domain $[z^{U-}, \omega^+]$. Note that the "excess" poverty headcount is the survival function: $\overline{F}(z) \equiv \Pr[x \geqslant z] = 1 - F(z)$. Hence condition (11) in Proposition 1 can alternatively be rendered: $\overline{F}^A(x) \leqslant \overline{F}^B(x) \quad \forall x \in [z^{U-}, \omega^+]$.

To illustrate numerically the conditions in Proposition 1, let consider distributions $A := (1, 4, 6, 9, 12, 14)$ and $B := (1, 4, 7, 8, 13, 14)$, and assume $z^{L+} = 5$ and $z^{U-} = 10$. Using Proposition 1, it can easily be seen that distribution A never shows more poverty than distribution B for all indices within Π^1 and all pairs of poverty line $\{z^L, Z^U\} \notin (z^{L+}, z^{U-})$ since

$F^A(x) = F^B(x)\ \forall x \in [\omega^-, 5] \cup [10, 12) \cup [13, \omega^+]$ but $F^A(x) > F^B(x)\ \forall x \in [12, 13)$. A similar situation is depicted by figure 1, which shows that the conditions from proposition 1 are fulfilled since distribution A's cdf is never above (below) distribution B's cdf for values of x lower (greater) than z^{L+} (z^{U-}).

Let $\boldsymbol{x}$ be a vector of values for the variable x and $\#(\boldsymbol{x})$ be the number of elements of $\boldsymbol{x}$. The following corollaries ensue directly from proposition 1:

Corollary 1. *There is a first-order stochastic dominance relationship between A and B $\forall P \in \Pi^1(z^{L+}, z^{U-})$ if $\exists \hat{\boldsymbol{x}} \in [z^{L+}, z^{U-}]^{\#(\hat{\boldsymbol{x}})}$ such that F^A and F^B cross only at the sole values of $\hat{\boldsymbol{x}}$ and $\#(\hat{\boldsymbol{x}})$ is an odd number.*

Proof. Obvious. ∎

Corollary 2. *If $z^{L+} = z^{U-} = \tilde{z}$, distribution A dominates distribution B at the first order $\forall P \in \Pi^1(z^{L+}, z^{U-})$ if and only if F^A and F^B cross only once and at $\tilde{z}$.*

Proof. Obvious. ∎

It is worth noting that Proposition 1 is reminiscent of famous results from the literature on risk (Rothschild and Stiglitz, 1970) and inequality (Atkinson, 1970) measurement as the distribution that shows more poverty also exhibits more weight at the tails of its distribution. However, Corollaries 1 and 2 show that dominance conditions are looser since risk and inequality dominance conditions are defined for the distributions of the variable x after normalization with respect to the mean, or for distributions with the same mean. As a consequence, robust results can only be obtained if the cumulative distribution functions cross once and only at the mean. Considering our framework, dominance relationships can be observed with any odd number of crossings as long as they happen outside the poverty domain. In the case of a single crossing, Corollary 2 states that the crossing value is no necessarily the average value of x but can be any other value that is consistent with admissible definitions of the maximum poverty domain.

Proposition 1 only provides a partial ordering for any pair of distributions defined on the domain Ω. In other words, the results with empirical implementations of the test are likely to be non-conclusive for a significant parts of the performed comparisons as it is possible to observe crossings of the cumulative distribution functions within the poverty domain. Thus it can be useful to add restrictions regarding the behaviour of poverty indices in the case of progressive transfers, an then to focus on members of the subclass Π_2.

While the dominance conditions for class Π_1 (Proposition 1) only require using a single function, namely the cumulative distribution function, the conditions for subclass Π_2 entail manipulating two different functions that accumulate gaps from the boundaries of the domain of x. Let $G(z) \equiv \int_{\omega^-}^{z} F(x)\,dx = \int_{\omega^-}^{z} (z - x)\,dF(x)$ and $\overline{G}(z) \equiv \int_{z}^{\omega^+} \overline{F}(x)\,dt = \int_{z}^{\omega^+} (x - z)\,d\overline{F}(x)$. The function $G(z)$ is known in the literature on poverty and wellbeing dominance as the absolute poverty gap index, and gives the mean value of the censored gaps $\max\{0, z - x\}$ observed in the population. The function $\overline{G}(z)$ does not average losses but excesses with respect to the value z, that is $\max\{0, x - z\}$. More precisely it is the product of the average excesses observed in the population with respect to threshold z times the part of that population whose level of x is larger than z. In practice, it deserves to be stressed that $\overline{G}(z)$ can be computed as $G(-z)$, that is the absolute poverty gap index for the variable $-x$. Then we show:

Proposition 2.

$$P_A\left(z^L, z^U\right) \leqslant P_B\left(z^L, z^U\right) \forall P \in \Pi^2\left(z^{L+}, z^{U-}\right) \tag{13}$$

$$\text{iff} \quad G^A(x) \leqslant G^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{14}$$

$$\text{and} \quad \overline{G}^A(x) \leqslant \overline{G}^B(x) \quad \forall x \in \left[z^{U-}, \omega^+\right]. \tag{15}$$

Proof. See appendix A.2 ■

The first part of the conditions presented in Proposition 2 is identical to the one suggested in Atkinson (1987) and Foster and Shorrocks (1988): for each value of x below z^{L+}, the value of the absolute poverty gap index should never be larger for population A than for population B. The second part considers the cumulative "excesses" and states that for poverty not to be higher in population A, the value of the absolute poverty gap index computed for the variable $-x$ should be lower than for population B for every value of x above the upper poverty line z^{U-}.

It is worth noting that, since we are dealing with sub-group additive poverty indices, we may distinguish two parts in the overall poverty level, that is the one corresponding to the presence of individuals within the bottom part of the poverty domain $[\omega^-, z^L]$ and the one corresponding to those people whose value of x is above the upper poverty line z^U. Overall poverty is consequently the sum of "loss" and "excess" poverty. Therefore we can focus on each group separately and then use only the corresponding condition in Propositions 1 and 2 to check whether a robust ordering can be obtained for the sole "loss" ("excess") poor when comparing two distributions. Using the example of distributions A and B page 8, we can see that both populations show the same level of "loss" poverty but that "excess" poverty is larger in a robust manner within population B.

3 The case of comparable deprivations

"Loss" and "excess" poverty may show very different causes and result in contrasted forms of wellbeing losses, but we may feel that both types may sometimes not deserve the same attention when estimating overall poverty. However, no a priori ordering of the situation of a "loss" poor and an "excess" poor can be performed directly as both person exhibit different values for the attribute x. To put additional assumptions regarding the extent of poverty when comparing both types of poor, it is thus necessary to move from variable x to a common space that makes deprivations easily comparable. As in Fisher and Spencer (1992) and Lambert and Zoli (2012), it may be useful to consider poverty indices defined with respect to distances (gaps) from the closest reference line for each individual and then

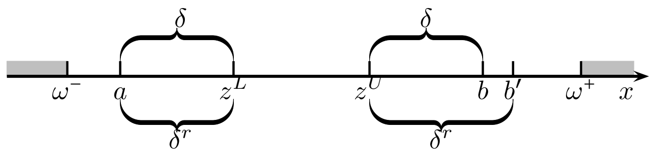

Figure 2: Comparability of the deprivations: absolute and relative gaps.

consider additional assumptions regarding the relative size of poverty for inviduals with different characteristics but showing the same gap.

3.1 Absolute gaps

Let $\delta \in \Re_+$ be accordingly defined as:

$$\delta := \begin{cases} z^L - x & \text{if } x \leqslant z^L, \\ 0 & \text{if } x \in (z^L, z^U), \text{ and} \\ x - z^U & \text{if } x \geqslant z^U, \end{cases} \tag{16}$$

Figure 2 shows the situation of two individuals, one is a "loss" poor with $x = a$ and the other one is an "excess" poor with $x = b$, so that $b = z^L + z^U - a$. It can easily be seen that these two individuals exhibit the same absolute gap δ, but if we assume that the situation of the "excess" poor cannot be regarded as severe as the situation of the "loss" poor, we should obtain $\pi\left(a; z^L, z^U\right) \geqslant \pi\left(b; z^L, z^U\right)$. If this behaviour can be regarded as reasonable for every potential value of δ, that is, given $x \leqslant z^L$ for all $\{x, z^L + z^U - x\} \subset \Omega$, we can then consider the following subclass of poverty indices:

$$\tilde{\Pi}^1(z^{L+}, z^{U-}) := \left\{ P \;\middle|\; \begin{array}{l} P \in \Pi^1(z^{L+}, z^{U-}) \\ \left|\pi^{(1)}(x, z^L, z^U)\right| \geqslant \pi^{(1)}(z^L + z^U - x, z^L, z^U) \;\; \forall x \leqslant z^L \text{ s.t. } (z^L + z^U - x) \in \Omega \end{array} \right\}. \tag{17}$$

The first condition in (17) states that members from $\tilde{\Pi}^1(z^{L+}, z^{U-})$ comply with the properties of indices from $\Pi^1(z^{L+}, z^{U-})$. The second condition defines the specificity of these indices and argue that the marginal gain from improving the situation of an "excess" poor is never greater than the marginal gain for a "loss" poor showing the same gap. It can easily be seen that, associated with the hypothesis of a zero poverty level at the deprivation lines, our additional assumption on the first-order derivatives of π is strictly equivalent to assuming that $\pi\left(x; z^L, z^U\right) \geqslant \pi\left(z^L + z^U - x; z^L, z^U\right)$. Members from $\tilde{\Pi}^1(z^{L+}, z^{U-})$ include for instance those from $FGT^A_{\beta,\alpha_L,\alpha_U}(z^L, z^U)$ for $\beta \in (0, 1)$ and $\alpha_L = \alpha_U$.

Considering different groups of poor people in a way that yields different individual poverty assessment for a given gap is not a new idea. Indeed, our framework is reminiscent of the literature related to monetary poverty comparisons with differences in needs associated with particular individuals' attributes, e.g. their household sizes (Bourguignon, 1989, Atkinson, 1992, Jenkins and Lambert, 1993, Chambaz and Maurin, 1998, Duclos and Makdissi, 2005, Lambert and Zoli, 2012). These studies show that the ordering power of stochastic dominance procedures can be increased when simple assumptions are made about the difference between the individual poverty indices corresponding to two different groups. Here, we suggest that, in many cases, a similar assumption can be made regarding the situation of the "loss" and the "excess" poor.

It deserves to be stressed that, for a "loss" value a and an "excess" value b of the considered wellbeing attribute to be directly comparable, both should thus show the same distance δ with respect to their corresponding poverty line. This point deserves to be stressed since poverty dominance checks are often performed to check the robustness with respect to changes in poverty lines. However, when considering gap dominance relationships, each couple (z^L, z^U) defines all the pairwise comparable values a and b within the "loss" and "excess" poverty domains. For instance, raising z^L by κ ($\kappa \in \Re_+$ with $\kappa < z^U - z^L$) while leaving z^U unchanged implies that the gap $\delta = x_2 - z^U$ does not make x_2 directly comparable with x_1 but with $x_1 + \kappa$. Consequently results obtained when comparing distributions A and B with the vector of poverty lines (z^L, z^U) may not hold when using the vector $(z^L + \kappa, z^U)$ as it refer to different sets of pairwise comparable values of the wellbeing attribute.

On the other hand, if z^L is increased by a given quantity κ and z^U decreased by the same amount (with, of course, $2\kappa < z^U - z^L$), the value of the gap for a and b would raise but the resulting gap $\delta - \kappa$ would still refer to the same values of the wellbeing attribute, leaving thus the correspondences between the "loss" poverty and "excess" poverty domains unchanged. With the assumption that a "loss" never yields less poverty than the corresponding "excess" given δ, one can thus consider the following conditions to be met to get ethically robust orderings considering members from this class of poverty indices $\tilde{\Pi}^1$:[7]

Proposition 3.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \quad \forall P \in \tilde{\Pi}^1(z^{L+}, z^{U-}),\; z^{L+} - z^L = z^U - z^{U-} = \kappa \tag{18}$$

$$\textit{and} \quad \kappa \in \left[0, \min\{z^{L+} - \omega^-, \omega^+ - z^{U-}\}\right) \tag{19}$$

$$\textit{iff} \quad F^A(x) \leqslant F^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{20}$$

$$\textit{and} \quad \overline{F}^A(x) + F^A(z^{L+} + z^{U-} - x) \leqslant \overline{F}^B(x) + F^B(z^{L+} + z^{U-} - x) \quad \forall x \in [z^{U-}, \omega^+]. \tag{21}$$

Proof. See appendix B.1. ■

Proposition 3 is a sequential dominance procedure in the spirit of those proposed in the aforementioned studies. First, condition (20) is the same as in Proposition 1 and states that the share of the population that experiences "loss" poverty, *i.e.* the neediest group, should be lower for population A than for population B at each value of $x \leqslant z^{L+}$ for poverty to be lower in the former population than in the latter population. The second condition does not make any difference between "loss" and "excess" gaps since both are brought together for a comparison of the cdf of gaps for each possible value of δ within the poverty

[7] A similar assumption is made in Lambert and Zoli (2012) for income poverty comparisons with group-specific poverty lines. As the authors consider gap-dominance relationships, they investigate the case of shifting all group specific poverty lines up by the same amount.

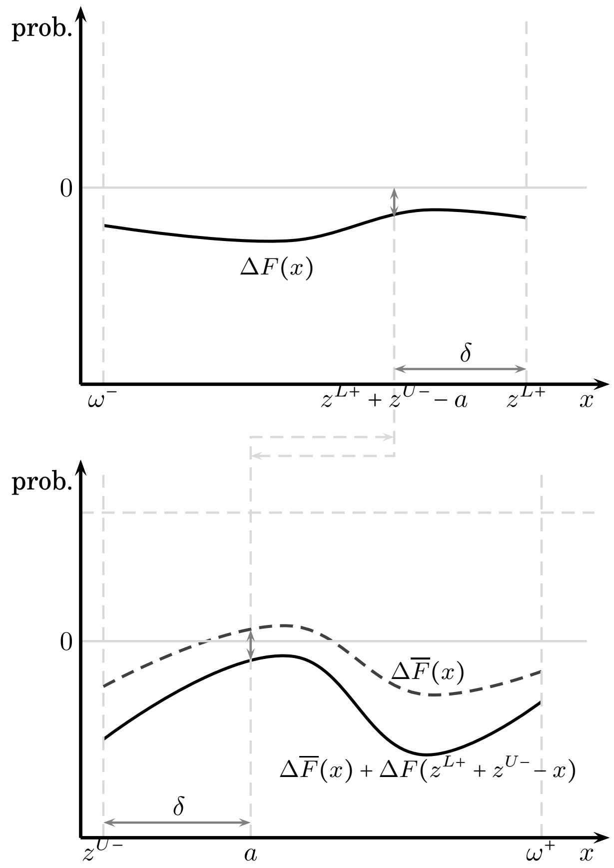

Figure 3: First order sequential gap dominance using Proposition 3.

domain (expressed in terms of gaps). Figure 3 illustrates these conditions. An interesting feature of the subclass $\tilde{\Pi}^1(z^{L+}, z^{U-})$ is that relatively poor performances of a population regarding "excess" poverty can be compensated by relatively good performances regarding the "loss" poor.

Let illustrate that point with an other example. Suppose distributions A and B are now respectively $(1,4,8,8,12)$ and $(1,2,7,7,11)$ still with $z^{L+} = 5$ and $z^{U-} = 10$. It can easily be seen that Proposition 1 does not hold since A exhibit less "loss" poverty than B but more "excess" poverty. However, if we suppose that a given gap δ yields more intense forms of poverty in the "loss" domain than in the "excess" domain, the two distributions can be ordered. Condition (20) is satisfied for each observed gap in the "loss" poverty domain. For the second condition, disregarding the nature of the gaps, we respectively obtain the following vectors of gaps $(0,0,1,2,4)$ and $(0,0,1,3,4)$ and it can then be seen that $F^A(x) + F^A(5+10-x) = F^B(x) + F^B(5+10-x)\ \forall x \in [\min\{\omega^-, 10+5-\omega^+\}, 2) \cup [4,5]$, but $F^A(x) + F^A(5+10-x) < F^B(x) + F^B(5+10-x)\ \forall x \in [2,4)$, so that condition (21) is also respected and we can conclude that A exhibits less poverty than B. It is also important to stress that the ordering is leaved unchanged if the lower and upper poverty lines are respectively decreased and raised by the same amount. For instance, if $z^L = z^{L+} - 1$ and $z^U = z^{U-} + 1$, we obtain the two vectors of gaps $(0,0,0,1,3)$ and $(0,0,0,2,3)$ and it can be seen that A still shows less poverty than distribution B whatever the precise functional form of P within $\tilde{\Pi}^1(z^{L+}, z^{U-})$.

It is worth noting that the sequential dominance conditions expressed in Proposition 3 differs from those traditional proposed in the sequential dominance literature—a notable exception is Bourguignon (1989)—as the poverty domain for the neediest group is not necessarily larger than the one for the less needy group. Indeed, if $z^{L+} - \omega^- \leqslant \omega^+ - z^{U-}$, the size of absolute gaps can be larger within the "excess" poverty domain than within the "loss" poverty domain, so that for values of $x \in (z^{L+} + z^{U-} - \omega^-, \omega^+]$ there are no possibility for "loss" poverty situations to compensate for "excess" poverty situations in condition 21.

As with the class of poverty indices Π^1, we also can assume that indices from Π^2 are more averse to inequality between the poor at the bottom of the distribution than at its upper tail. We then consider the class $\tilde{\Pi}^2$ such that:

$$\tilde{\Pi}^2(z^{L+}, z^{U-}) := \left\{ P \,\middle|\, \begin{array}{l} P \in \tilde{\Pi}^1(z^{L+}, z^{U-}) \cup \Pi^2(z^{L+}, z^{U-}) \\ \pi^{(2)}(x, z^L, z^U) \geqslant \pi^{(2)}(z^L + z^U - x, z^L, z^U) \ \forall x \leqslant z^L \text{ s.t. } \{x, z^L + z^U - x\} \subset \Omega \end{array} \right\}. \tag{22}$$

The first condition in (22) states that members from $\tilde{\Pi}^2(z^{L+}, z^{U-})$ forms a common subclass of both $\tilde{\Pi}^1(z^{L+}, z^{U-})$ and $\Pi^2(z^{L+}, z^{U-})$. The second line in (22) states that the marginal gains from improving the situation of a "loss" poor decrease more rapidly than for the "excess" poor.

Proposition 4.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \quad \forall P \in \tilde{\Pi}^2(z^{L+}, z^{U-}),\ z^{L+} - z^L = z^U - z^{U-} = \kappa \tag{23}$$

$$\textit{and} \quad \kappa \in \left[0, \min\{z^{L+} - \omega^-, \omega^+ - z^{U-}\}\right) \tag{24}$$

$$\textit{iff} \quad G^A(x) \leqslant G^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{25}$$

$$\textit{and} \quad \overline{G}^A(x) + G^A(z^{L+} + z^{U-} - x) \leqslant \overline{G}^B(x) + G^B(z^{L+} + z^{U-} - x) \quad \forall x \in [z^{U-}, \omega^+]. \tag{26}$$

Proof. See appendix B.2. ■

While Propositions 3 and 4 allow for a large set of choices for the poverty lines (z^L, z^U), we may feel that the conditions on the linkages between z^L and z^U given z^{L+} and z^{U-} are much too restrictive as they do not make it possible to chose freely the vector of poverty lines within the set $[z^{L-}, z^{L+}] \times [z^{U-}, z^{U+}]$ of admissible couples of poverty lines. If one desires to get such flexibility, it is then necessary to consider the following propositions:

Proposition 5.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \quad \forall P \in \tilde{\Pi}^1(z^{L+}, z^{U-}),\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}] \tag{27}$$

$$\textit{iff} \quad F^A(x) \leqslant F^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{28}$$

$$\textit{and} \quad \overline{F}^A(x) + F^A(z^{L+} + z^{U-} - x) \leqslant \overline{F}^B(x) + F^B(z^{L+} + z^{U-} - x) \tag{29}$$

$$\forall x \in [z^{U-}, \omega^+],\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}].$$

Proposition 6.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \quad \forall P \in \tilde{\Pi}^2(z^{L+}, z^{U-}),\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}] \tag{30}$$

$$\textit{iff} \quad G^A(x) \leqslant G^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{31}$$

$$\textit{and} \quad \overline{G}^A(x) + G^A(z^{L+} + z^{U-} - x) \leqslant \overline{G}^B(x) + G^B(z^{L+} + z^{U-} - x) \tag{32}$$
$$\forall x \in [z^{U-}, \omega^+],\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}].$$

Proof. See appendices B.1 and B.2. ∎

While such conditions provide more robust conditions than those given by Propositions 3 and 4, it can easily be seen that they are very computationally intensive. From a practical point of view, it is worth noting that, since Proposition 5 (Propostion 6) is a generalization of Proposition 3 (Propostion 4), the conditions in the former will never be met if those of the latter are not fulfilled. Checking first the easily implementable conditions (20) and (21) is thus a good way of saving ourself from tedious calculus.

However, it can be shown that conditions (29) and (32) can expressed in a different manner that make it possible implement it more easily in the spirit of Bourguignon (1989). Let $\varphi_1(x)$ be the maximum value of the difference $F^A(y) - F^B(y)$ for a given value of $x \in [z^{U-}, \omega^+]$ where y denotes the value of the wellbeing attribute that is likely to correspond to the same gap within the "loss" poverty domain that x within the "excess" poverty domain, that is:

$$\varphi_1(x) = \max_{y \in \Lambda(x)} F^A(y) - F^B(y), \tag{33}$$

where $\Lambda(x) = \left[\, \max\{\omega^-, z^{L-} + z^{U-} - x\}, z^{L+} - \max\{0, x - z^{U+}\}\right]$. In the same spirit, we define $\varphi_2(x)$ as:

$$\varphi_2(x) = \max_{y \in \Lambda(x)} \int_{\omega^-}^{y} F^A(t) - F^B(t)\, dt. \tag{34}$$

Propositions 5 and 6 can then be alternatively expressed as:

Proposition 7.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U)\ \forall P_\delta \in \tilde{\Pi}^1(z^{L+}, z^{U-}),\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}] \tag{35}$$

$$\textit{iff} \quad F^A(x) \leqslant F^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{36}$$

$$\textit{and} \quad \overline{F}^A(x) - \overline{F}^B(x) + \varphi_1(x) \leqslant 0 \quad \forall x \in [z^{U-}, \omega^+]. \tag{37}$$

Proposition 8.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U)\ \forall P_\delta \in \tilde{\Pi}^2(z^{L+}, z^{U-}),\ z^L \in [z^{L-}, z^{L+}],\ \textit{and}\ z^U \in [z^{U-}, z^{U+}] \tag{38}$$

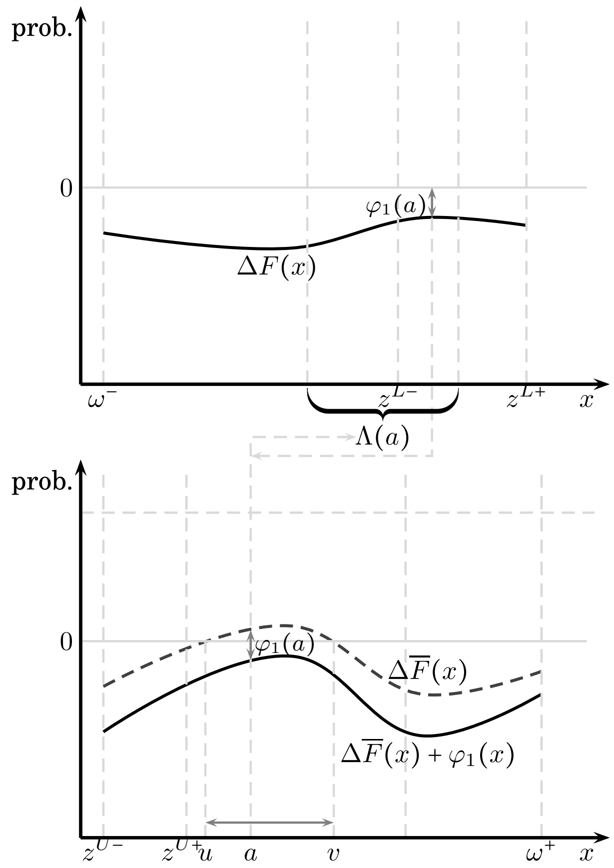

Figure 4: First order sequential gap dominance using Proposition 7.

$$\textit{iff} \quad G^A(x) \leqslant G^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{39}$$

$$\textit{and} \quad \overline{G}^A(x) - \overline{G}^B(x) + \varphi_2(x) \leqslant 0 \quad \forall x \in [z^{U-}, \omega^+]. \tag{40}$$

Proof. See appendix B.3. ∎

Figure 4 illustrates Proposition 7. The upper part illustrates the first step of the procedure. The curve plots the difference $F^A(x) - F^B(x)$ over the maximum "loss" poverty domain. Condition (36) is fulfilled since the curve systematically returns negative values over the interval $[\omega^-, z^{L+}]$. Both the lower and upper panels are needed for the second step of the procedure. The dashed curve represents the difference $\overline{F}^A(x) - \overline{F}^B(x)$ over the maximum "excess" poverty domain. As condition (36) is respected, $\varphi_1(x)$ is non-positive and condition (37) will necessarily be satisfied when the dashed curve is below the horizontal line. So, condition (37) could possibly not be respected when the dashed curve is above the horizontal lines, that is for values of $x \in (u, v)$. Then for each value a within this interval, we first look at the corresponding interval $\Lambda(a)$ in the "loss" poverty domain and consider the values of $F^A(x) - F^B(x)$ for each value within $\Lambda(a)$. The largest values corresponds to $\varphi_1(a)$ and is added to $\overline{F}^A(x) - \overline{F}^B(x)$ in the lower panel. The continuous black curve in the lower part of Figure 4 thus plots $\overline{F}^A(x) - \overline{F}^B(x) + \varphi_1(x)$ for each value within the maximum "excess" poverty domain and it can be seen that condition (37) is fulfilled since the curve is always below the zero horizontal line. Consequently, it can be said that there is more poverty in distribution B than in distribution A once we have presumed that "loss" poverty was more a matter of concern that "excess" poverty.

Let illustrate now numerically the proposed algorithm with a simple example. We assume $(\omega^-, z^{L-}, z^{L+}, z^{U-}, z^{U+}, \omega^+) = (0, 8, 10, 15, 20, 30)$, $A = (3, 9, 12, 12, 12, 12, 17, 18)$, and $B = (1, 1, 2, 8, 12, 12, 16, 24)$. We can observe that condition (11) is fulfilled $\forall x \in [0, 10]$, but (12) does not hold for $x \in (16, 17]$ so that Proposition 1 does not hold. Since condition (21) is met (Proposition 3 can thus be applied), it is worth considering condition (37). As $\overline{F}^A(x) - \overline{F}^B(x) > 0$ only for $x \in (16, 17]$ it is not necessary compute $\varphi_1(x)$ for values outside this interval. For values of x within $(16, 17]$ it can be checked that we have to look for the highest value of $F^A(x) - F^B(x)$ within $\bigcup_{x \in (16,17]} \Lambda(x) = \Lambda(17) = [5, 10)$. We then find $(\overline{F}^A(17) - \overline{F}^B(17)) + \varphi_1(17) = \frac{1}{8} - \frac{2}{8} < 0$. Condition (37) is thus satisfied as satisfied since $\Delta\overline{F}(x) + \varphi_1(x) \leqslant 0 \ \forall x \in (15, 30]$. Consequently we can argue that there isn't more poverty in population A than in population B whatever the poverty index from $\Theta^1(z^{L+}, z^{U-})$ and the couple of poverty lines within the subset $[8, 10] \times [15, 20]$.

It is worth noting that the power of Propositions 7 and 8 depends heavily on the chosen values for the minimum and maximum poverty lines. In particular, as the probability of satisfying condition (37) depends on the width of $\Lambda(x)$, the ordering power of the two propositions should decrease as the ranges for z^L and z^U increase. For instance, in our last example, we observed $\Lambda(21) = [2, 9]$ for $z^L \in [8, 10]$ and $z^U \in [15, 20]$. With $z^L \in [9, 10]$ and $z^U \in [15, 17]$, then $\Lambda(21)$ would have shrunk to $[3, 6]$ and thus have lowered the probability to get $\overline{F}^A(21) - \overline{F}^B(21)) + \varphi^1(21) > 0$.

3.2 Relative gaps

Up to now, we have considered social poverty indices whose individual poverty indices are based on absolute deviations from the poverty lines. However, a usual practice is to regard deprivations in terms of relative gaps, e.g. as in the measures W_β and $FGT_{\beta,\alpha_L,\alpha_U}$ proposed in equations (6) and (7). That is, we can use δ^r such that:

$$\delta^r := \begin{cases} \frac{z^L - x}{z^L - \omega^-} & \text{if } x \leqslant z^L, \\ 0 & \text{if } x \in (z^L, z^U), \text{ and} \\ \frac{x - z^U}{\omega^+ - z^U} & \text{if } x \geqslant z^U. \end{cases} \tag{41}$$

When $z^L - \omega^- \neq \omega^+ - z^U$, assessing poverty would not be affected by a change from absolute gaps to relative gaps, but in other cases as illustrated in figure 2, such a change affect poverty orderings when additional assumption are made regarding the relative contribution of "loss" and "excess" poverty to overall poverty. Using relative gaps δ^r, instead of absolute gaps δ, when performing the first-order and second-order dominance checks described in Proposition 1 and 2, does not change the results. Yet different results may ensue for Propositions 3 and 4 since relative gaps do not correspond to the same values of absolute gaps when $z^L - \omega^- \neq \omega^+ - z^U$. Moreover, dominance results with relative gaps are likely to be contingent to choices for the values of ω^- or ω^+.

If comparability of the two forms of poverty is based on relative gaps, we then have consider the following subclasses of poverty indices:

$$\tilde{\Pi}_r^1(z^{L+}, z^{U-}) := \left\{ P \,\middle|\, \begin{array}{l} P \in \Pi^1(z^{L+}, z^{U-}) \\ \left|\pi^{(1)}(x, z^L, z^U)\right| \geqslant \pi^{(1)}\left(z^U + \frac{z^L - x}{z^L - \omega^-}(\omega^+ - z^U), z^L, z^U\right) \;\forall x \leqslant z^L \end{array} \right\}. \tag{42}$$

$$\tilde{\Pi}_r^2(z^{L+}, z^{U-}) := \left\{ P \,\middle|\, \begin{array}{l} P \in \tilde{\Pi}^1(z^{L+}, z^{U-}) \cup \Pi^2(z^{L+}, z^{U-}) \\ \pi^{(2)}(x, z^L, z^U) \geqslant \pi^{(2)}\left(z^U + \frac{z^L - x}{z^L - \omega^-}(\omega^+ - z^U), z^L, z^U, \right) \;\forall x \leqslant z^L \end{array} \right\}. \tag{43}$$

The counterparts of Proposition 3 and 4 for relative gaps are then:

Proposition 9.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \quad \forall P \in \tilde{\Pi}^1(z^{L+}, z^{U-}),\; \frac{z^{L+} - z^L}{z^{L+} - \omega^-} = \frac{z^U - z^{U-}}{\omega^+ - z^{U-}} = \kappa \in [0,1) \tag{44}$$

$$\textit{iff} \quad F^A(x) \leqslant F^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{45}$$

$$\textit{and} \quad \overline{F}^A(x) + F^A\left(z^{L+} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^L - \omega^-)\right) \leqslant \overline{F}^B(x) + F^B\left(z^{L+} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^L - \omega^-)\right)$$
$$\forall x \in [z^{U-}, \omega^+]. \tag{46}$$

Proposition 10.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \quad \forall P \in \tilde{\Pi}^2(z^{L+}, z^{U-}),\; \frac{z^{L+} - z^L}{z^{L+} - \omega^-} = \frac{z^U - z^{U-}}{\omega^+ - z^{U-}} = \kappa \in [0,1) \tag{47}$$

$$\textit{iff} \quad G^A(x) \leqslant G^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{48}$$

$$\textit{and} \quad \overline{G}^A(x) + G^A\left(z^{L+} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^L - \omega^-)\right) \leqslant \overline{G}^B(x) + G^B\left(z^{L+} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^L - \omega^-)\right)$$
$$\forall x \in [z^{U-}, \omega^+]. \tag{49}$$

Proof. See appendix B.1. ■

Let $\varphi_k^r(x)$, $k = 1, 2$, be the counterpart of $\varphi^k(x)$ with relative gaps. In the case $\varphi_1^r(x)$ we then obtain:

$$\varphi_1^r(x) = \max_{y \in \Lambda^r(x)} F^A(y) - F^B(y), \tag{50}$$

where $\Lambda^r(x) = \left[z^{L-} + \frac{z^{U-} - x}{\omega^+ - z^{U-}}(z^{L-} - \omega^-), z^{L+} + \min\left\{0, \frac{z^{U+} - x}{\omega^+ - z^{U+}}(z^{L+} - \omega^-)\right\}\right]$.

Proposition 11.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \; \forall P \in \tilde{\Pi}_r^1(z^{L+}, z^{U-}),\; z^L \in [z^{L-}, z^{L+}],\; \textit{and}\; z^U \in [z^{U-}, z^{U+}] \tag{51}$$

$$\textit{iff} \quad F^A(x) \leqslant F^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{52}$$

$$\textit{and} \quad \overline{F}^A(x) - \overline{F}^B(x) + \varphi_1^r(x) \leqslant 0 \quad \forall x \in [z^{U-}, \omega^+]. \tag{53}$$

Proposition 12.

$$P_A(z^L, z^U) \leqslant P_B(z^L, z^U) \ \forall P \in \tilde{\Pi}_r^2(z^{L+}, z^{U-}), \ z^L \in [z^{L-}, z^{L+}], \ \textit{and} \ z^U \in [z^{U-}, z^{U+}] \tag{54}$$

$$\textit{iff} \quad G^A(x) \leqslant G^B(x) \quad \forall x \in \left[\omega^-, z^{L+}\right] \tag{55}$$

$$\textit{and} \quad \overline{G}^A(x) - \overline{G}^B(x) + \varphi_2^r(x) \leqslant 0 \quad \forall x \in [z^{U-}, \omega^+]. \tag{56}$$

4 Statistical inference

In empirical applications we estimate the following discrete counterpart of equation (4):

$$P(z^L, z^U) = \frac{1}{N}\sum_{n=1}^{N} \pi(x_n, z^L, z^U), \tag{57}$$

where N is the sample size and x_n is the value of x for individual n. Now, generally the functions π are likely to be different for "loss" and "excess" poverty, just as in the examples of (6) and (7). Hence we can write equation (57) as the sum of two distinct functions π, each multiplied by poverty identification functions:

$$\begin{aligned} P(z^L, z^U) &= \frac{1}{N}\sum_{n=1}^{N}[\pi(x_n, z^L, z^U)\mathbb{I}(x_n < z^L) + \pi(x_n, z^L, z^U)\mathbb{I}(x_n > z^U)] \\ &= \frac{1}{N}\sum_{n=1}^{N}[\pi(x_n, z^L, z^U)\mathbb{I}(x_n < z^L)] + \frac{1}{N}\sum_{n=1}^{N}[\pi(x_n, z^L, z^U)\mathbb{I}(x_n > z^U)], \end{aligned} \tag{58}$$

where $\mathbb{I}(test)$ is an identification function returning 1 if $test$ is fulfilled and 0 otherwise. Now the standard error corresponding to expression (58) of P is going to depend on the estimates $\hat{\sigma}_L$ and $\hat{\sigma}_U$ of the standard errors of the two averages on the right-hand side plus a negative covariance term. This covariance is negative because whenever $x_n < z^L$ then it is not the case that $x_n > z^U$, and viceversa. After some straightforward manipulations the variance of P is thus:

$$V(P) = \frac{\hat{\sigma}_L^2 + \hat{\sigma}_U^2 - 2P_L P_U}{N}, \tag{59}$$

where:

$$P_L := \frac{1}{N}\sum_{n=1}^{N}[\pi(x_n, z^L, z^U)\mathbb{I}(x_n < z^L)], \tag{60}$$

$$P_U := \frac{1}{N}\sum_{n=1}^{N}[\pi(x_n, z^L, z^U)\mathbb{I}(x_n > z^U)], \tag{61}$$

$$\hat{\sigma}_L^2 := \frac{1}{N}\left(\sum_{n=1}^{N}\pi(x_n, z^L, z^U)^2\mathbb{I}(x_n < z^L)\right) - P_L^2, \tag{62}$$

$$\hat{\sigma}_U^2 := \frac{1}{N}\left(\sum_{n=1}^{N}\pi(x_n, z^L, z^U)^2\mathbb{I}(x_n > z^U)\right) - P_U^2. \tag{63}$$

The formulas can be used for stochastic dominance tests applicable to the conditions derived in this paper. The testing procedures proposed in Kaur et al. (1994), Davidson and Duclos (2000) and Davidson and Duclos (2012) are appealing starting point as they are based on rival hypothesess that make it possible to conclude in a statistically robust manner whether a distribution dominates an other distribution for a given class of social evaluation functions. Basically, the test consists in a first step to oppose for each value of x within the poverty domain the following hypothesis.

$$\begin{cases} H_0 : \Delta S(x) \geqslant 0, \\ H_1 : \Delta S(x) < 0. \end{cases} \tag{64}$$

where $\Delta S(x)$ is the considered criteria, for instance $F^A(x) - F^B(x)$ in the case of Proposition 1. Non-dominance of distribution A with respect to distribution B occurs when H_0 cannot be rejected. As the functions used for the dominance criteria are basically the average value of individual functions, their value is asymptotically distributed and H_0 can be tested using a simple t-test. Since the test has to be performed over the whole poverty domain, it can be concluded that distribution A dominates distribution B in a statistically significant manner if H_0 is rejected for each value of x within the poverty domain at the chosen confidence level. The test statistics for the whole procedure suggested by Kaur et al. (1994) is consequently:

$$t_{\min} = \min \left\{ \frac{\Delta \hat{S}(x)}{\sqrt{\hat{V}\big(S^A(x)\big) + \hat{V}\big(S^B(x)\big)}} \middle| x \in [\omega^-, z^{L+}] \cup [z^{U-}, \omega^+] \right\} \tag{65}$$

where N^Y is the sample size of distribution Y. Dominance is thus observed if $t_{\min}$ is less than the corresponding quantile of the Student distribution for the given confidence level.

In spite of its appeal, the procedure is empirically not tractable unless distributions are censored at their tail as noted by Davidson and Duclos (2012). Indeed most observed distributions are likely to show $F(\omega^-) = 0$ or $\overline{F}(\omega^+) = 0$. Consequently it is highly probable to obtain $\Delta S(x) = 0$ so that estimating $t_{\min}$ systematically results in the non-rejection of H_0. As noted by Davidson and Duclos (2012), while censoring may *a priori* be at odd with the core axiomatic framework of poverty measurement, especially strong versions of MON, there are valuable reasons for performing such censoring. From a practical point of view, censoring may be necessary as stochastic dominance procedures are highly sensitive to the presence of outliers: small measurement errors at the tails of the distribution may yield a non-dominance result though dominance should objectively be concluded. From an ethical point of view, it can be said that there are some thresholds at the two tails of Ω under and above which deprivation is total. For instance, let imagine two overweighted persons whose weight prevent them from moving by themselves and so show limited social life and high risks of premature death. If these two individuals are plainly identical except that the first is 10kg lighter than the second, hence resulting in a lower value of the BMI, we could reasonably argue that the BMI difference is not worth assuming even a marginal difference with respect to their individual poverty evaluation. Such individuals ought not to be dropped from the compared sample but to be treated as if they were exactly at the corresponding threshold of complete deprivation.

5 Empirical illustration: Health poverty in Bangladesh

TO DO!

6 Conclusion

Assessing human progresses with respect to health outcomes has a long history and the recent widespread recognition of the multidimensional nature of poverty both inside and outside the academic field has made possible the use of poverty measurement tools to assess the extent of deprivations within the health dimension of wellbeing. However, contrary to traditional uses in monetary poverty, health indicators are likely to be related to wellbeing in a non-monotonic manner, so that individuals may suffer from either too low or too high levels of such variables. Providing a synthetic index for health-related illfare that can fully take into account the dual burden of undernutrition and obesity is thus a challenging issue that deserves consideration. As stressed by Tanumihardjo et al. (2007, p.1969) "historically, national and international food programs have focused on providing communities with staple crops to meet the energy needs of people so that they can lead productive lives. Today, this practice continues despite the health-related consequences of resulting obesity. Now is the time to reevaluate these practices and change policies to promote healthful, nutritionally dense eating practices rather than harmful energy-dense eating practices." It is thus of prime importance not to focus on the sole aspects of undernutrition for the assessment of nutrition policies as the social costs for alleviating undernutrition problems may sometimes outweigh the gains.

In the present paper, we proposed some alterations of traditional poverty measurement axioms in order to propose an health-related illfare assessment that is consistent with our non-monotonic wellbeing relationships. Furthermore, we provide dominance criteria that make it possible to assess the ethical robustness of health-related illfare orderings considering broad classes of illfare indices based on some reasonable assumptions and admissible ranges for the deprivation lines. Further developments imply of course the development of dominance technique when such non-monotonic relationships occur in a multidimensional framework, for instance when information on income, education or access to basic services are added to health variables in order to get a more comprehensive picture of illfare.

Appendices

A Proof of dominance conditions

A.1 Proposition 1

Let $\Delta P \equiv P_A - P_B$ be the difference between the statistics (e.g. P, or F) of populations A and B. Then note that equation (4) for the difference ΔP can be expressed as:

$$\Delta P(z^L, z^U) = \int_{\omega^-}^{z^L} \pi(x; z^L, z^U)\Delta f(x)\, dx + \int_{z^U}^{\omega^+} \pi(x; z^L, z^U)\Delta f(x)\, dx. \tag{66}$$

where $f : \Omega \to [0,1]$ is the density function. Integrating by parts each term in equation (66), we obtain:

$$\begin{aligned}\Delta P(z^L, z^U) &= \left[\pi(x; z^L, z^U)\Delta F(x)\right]_{\omega^-}^{z^L} - \int_{\omega^-}^{z^L} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\, dx \\ &+ \left[\pi(x; z^L, z^U)\Delta F(x)\right]_{z^U}^{\omega^+} - \int_{z^U}^{\omega^+} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\, dx. \end{aligned}\tag{67}$$

Since $\pi(z^L; z^L, z^U) = \pi(z^U; z^L, z^U) = 0$ and $\Delta F(\omega^-) = \Delta F(\omega^+) = 0$, we obtain:

$$\Delta P(z^L, z^U) = -\int_{\omega^-}^{z^L} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\, dx - \int_{z^U}^{\omega^+} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\, dx. \tag{68}$$

The rest of the proof follows by inspection.

A.2 Proposition 2

Noting that in univariate settings: $F(x) = 1 - \overline{F}(x)$ and therefore: $\Delta F(x) = -\Delta\overline{F}(x)$; we integrate equation (68) by parts, expressing the first right-hand side element in terms of F and the second right-hand side element in terms of $\overline{F}$. Keeping in mind that $\frac{\partial \overline{G}}{\partial x} = -\overline{F}(x)$, this yields:

$$\Delta P(z^L, z^U) = -\int_{\omega^-}^{z^L} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\, dx - \int_{z^U}^{\omega^+} \pi^{(1)}(x; z^L, z^U)\left(-\Delta\overline{F}(x)\right)\, dx, \tag{69}$$

$$\begin{aligned} &= -\left[\pi^{(1)}(x; z^L, z^U)\Delta G(x)\right]_{\omega^-}^{z^L} + \int_{\omega^-}^{z^L} \pi^{(2)}(x; z^L, z^U)\Delta G(x)\, dx \\ &- \left[\pi^{(1)}(x; z^L, z^U)\Delta\overline{G}(x)\right]_{z^U}^{\omega^+} + \int_{z^U}^{\omega^+} \pi^{(2)}(x; z^L, z^U)\Delta\overline{G}(x)\, dx, \end{aligned}\tag{70}$$

$$= \int_{\omega^-}^{z^L} \pi^{(2)}(x; z^L, z^U)\Delta G(x)\, dx + \int_{z^U}^{\omega^+} \pi^{(2)}(x; z^L, z^U)\Delta\overline{G}(x)\, dx. \tag{71}$$

since $\pi^{(1)}(z^L; z^L, z^U) = \pi^{(1)}(z^U; z^L, z^U) = 0$ and $\Delta G(\omega^-) = \Delta\overline{G}(\omega^+) = 0$. The rest of the proof follows by inspection.

B Proof of sequential dominance conditions

B.1 Proof of Propositions 3 and 5

Considering $\pi^{(x)}(z^L + z^U - x; z^L, z^U) = -\pi^{(1)}(z^L + z^U - x; z^L, z^U)$ it can first be seen that (68) can be rewritten in the following way:

$$\begin{aligned}
\Delta P(z^L, z^U) &= -\int_{\omega^-}^{z^L} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\,dx \\
&\quad + \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(x)}(z^L + z^U - x; z^L, z^U)\Delta\overline{F}(z^L + z^U - x)\,dx, && (72)\\
&= -\int_{\omega^-}^{z^L} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\,dx \\
&\quad - \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\Delta\overline{F}(z^L + z^U - x)\,dx. && (73)
\end{aligned}$$

In the case $z^{L+} - \omega^- \geqslant \omega^+ - z^{U-}$, equation (73) can be expressed as:

$$\begin{aligned}
\Delta P(z^L, z^U) &= -\int_{\omega^-}^{z^L+z^U-\omega^+} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\,dx \\
&\quad - \int_{z^L+z^U-\omega^+}^{z^L} \left(\pi^{(1)}(x; z^L, z^U) + (1-1)\pi^{(1)}(z^L + z^U - x; z^L, z^U)\right)\Delta F(x)\,dx \\
&\quad + \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\Delta\overline{F}(z^L + z^U - x)\,dx, && (74)\\
&= -\int_{\omega^-}^{z^L+z^U-\omega^+} \pi^{(1)}(x; z^L, z^U)\Delta F(x)\,dx \\
&\quad - \int_{z^L+z^U-\omega^+}^{z^L} \left(\pi^{(1)}(x; z^L, z^U) + \pi^{(1)}(z^L + z^U - x; z^L, z^U)\right)\Delta F(x)\,dx \\
&\quad + \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\left(\Delta\overline{F}(z^L + z^U - x) + \Delta F(x)\right)\,dx. && (75)
\end{aligned}$$

By assumption, $\pi^{(1)}(x; z^L, z^U) + \pi^{(1)}(z^L + z^U - x; z^L, z^U) \leqslant 0\ \forall x \in [z^L + z^U - \omega^+, z^L]$. The rest of the proof follows by inspection.

In the case $z^{L+} - \omega^- \leqslant \omega^+ - z^{U-}$, equation (73) can be expressed as:

$$\begin{aligned}
\Delta P(z^L, z^U) &= -\int_{\omega^-}^{z^L} \left(\pi^{(1)}(x; z^L, z^U) + (1-1)\pi^{(1)}(z^L + z^U - x; z^L, z^U)\right)\Delta F(x)\,dx \\
&\quad + \int_{z^L+z^U-\omega^+}^{\omega^-} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\Delta\overline{F}(z^L + z^U - x)\,dx \\
&\quad + \int_{\omega^-}^{z^L} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\Delta\overline{F}(z^L + z^U - x)\,dx, && (76)\\
&= -\int_{\omega^-}^{z^L} \left(\pi^{(1)}(x; z^L, z^U) + \pi^{(1)}(z^L + z^U - x; z^L, z^U)\right)\Delta F(x)\,dx \\
&\quad + \int_{z^L+z^U-\omega^+}^{\omega^-} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\Delta\overline{F}(z^L + z^U - x)\,dx \\
&\quad + \int_{\omega^-}^{z^L} \pi^{(1)}(z^L + z^U - x; z^L, z^U)\left(\Delta\overline{F}(z^L + z^U - x) + \Delta F(x)\right)\,dx. && (77)
\end{aligned}$$

By assumption, $\pi^{(1)}(x; z^L, z^U) + \pi^{(1)}(z^L + z^U - x; z^L, z^U) \leqslant 0\ \forall x \in [\omega^-, z^L]$. The rest of the proof follows by inspection.

B.2 Proof of Propositions 4 and 6

Considering members from $\tilde{\Pi}^2(z^{L+}, z^{U-})$, we first can rewrite equation (71) as:

$$\Delta P(z^L, z^U) = \int_{\omega^-}^{z^L} \pi^{(2)}(x; z^L, z^U)\Delta G(x)\,dx + \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(2)}(z^L+z^U-x; z^L, z^U)\Delta\overline{G}(z^L+z^U-x)\,dx. \tag{78}$$

In the case $z^{L+} - \omega^- \geqslant \omega^+ - z^{U-}$, equation (78) can be expressed as:

$$\begin{aligned}\Delta P(z^L, z^U) &= \int_{\omega^-}^{z^L+z^U-\omega^+} \pi^{(2)}(x; z^L, z^U)\Delta G(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{z^L} \left(\pi^{(2)}(x; z^L, z^U) + (1-1)\pi^{(2)}(z^L+z^U-x; z^L, z^U)\right)\Delta G(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(2)}(z^L+z^U-x; z^L, z^U)\Delta\overline{G}(z^L+z^U-x)\,dx, \qquad (79) \\ &= \int_{\omega^-}^{z^L+z^U-\omega^+} \pi^{(2)}(x; z^L, z^U)\Delta G(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{z^L} \left(\pi^{(2)}(x; z^L, z^U) - \pi^{(2)}(z^L+z^U-x; z^L, z^U)\right)\Delta G(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{z^L} \pi^{(2)}(z^L+z^U-x; z^L, z^U)\left(\Delta\overline{G}(z^L+z^U-x) + \Delta G(x)\right)\,dx. \qquad (80)\end{aligned}$$

By assumption, $\pi^{(2)}(x; z^L, z^U) + \pi^{(2)}(z^L+z^U-x; z^L, z^U) \geqslant 0\ \forall x \in [z^L+z^U-\omega^+, z^L]$. The rest of the proof follows by inspection.

In the case $z^{L+} - \omega^- \leqslant \omega^+ - z^{U-}$, equation (78) can be expressed as:

$$\begin{aligned}\Delta P(z^L, z^U) &= \int_{\omega^-}^{z^L} \left(\pi^{(2)}(x; z^L, z^U) + (1-1)\pi^{(2)}(z^L+z^U-x; z^L, z^U)\right)\Delta G(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{\omega^-} \pi^{(2)}(z^L+z^U-x; z^L, z^U)\Delta\overline{G}(z^L+z^U-x)\,dx \\ &+ \int_{\omega^-}^{z^L} \pi^{(2)}(z^L+z^U-x; z^L, z^U)\Delta\overline{G}(z^L+z^U-x)\,dx, \qquad (81) \\ &= \int_{\omega^-}^{z^L} \left(\pi^{(2)}(x; z^L, z^U) - \pi^{(2)}(z^L+z^U-x; z^L, z^U)\right)\Delta G(x)\,dx \\ &+ \int_{z^L+z^U-\omega^+}^{\omega^-} \pi^{(2)}(z^L+z^U-x; z^L, z^U)\Delta\overline{G}(z^L+z^U-x)\,dx \\ &+ \int_{\omega^-}^{z^L} \pi^{(2)}(z^L+z^U-x; z^L, z^U)\left(\Delta\overline{G}(z^L+z^U-x) + \Delta G(x)\right)\,dx. \qquad (82)\end{aligned}$$

By assumption, $\pi^{(2)}(x; z^L, z^U) - \pi^{(2)}(z^L+z^U-x; z^L, z^U) \geqslant 0\ \forall x \in [\omega^-, z^L]$. The rest of the proof follows by inspection.

B.3 Proof of Propositions 7 and 8

The proof is inspired from Lambert and Zoli (2005).

Let first consider the case $x \in [z^{U-}, z^{U+}]$. It can then easily be seen that the largest potential value of delta is $\delta^+ = x - z^{U-}$ while the lowest is $\delta^- = 0$. For a given set of poverty lines z^L, z^U, the value y within the "loss" poverty domain that yields the same gap as x is $z^L - \delta$. Since $z^L \in [z^{L-}, z^{L+}]$, we then have $y \in [z^{L-} - \delta, z^{L+} - \delta]$ for a given value of δ. Taking into account that $\delta \in [0, x - z^{U-}]$, we then have $y \in [z^{L-} + z^{U-} - x, z^{L+}]$. Finally, as we should have $y \geqslant \omega^-$ but may observe $z^{L-} + z^{U-} - x < \omega^-$, we find $y \in \big[\max\{\omega^-, z^{L-} + z^{U-} - x\}, z^{L+}\big]$.

Now, let have a look at the case $x \in [z^{U+}, \omega^+]$. Potential values of δ are then $\delta^+ = x - z^{U-}$ and $\delta^- = x - z^{U+}$. For a given value of x, taking the variability of z^L and z^U into account, we obtain $y \in [z^{L-} - \delta^+, z^{L+} - \delta^-] = [z^{L-} + z^{U-} - x, z^{L+} + z^{U+} - x]$. Once again, we have to observe $y \geqslant \omega^-$ but it is possible to have $z^{L-} + z^{U-} - x < \omega^-$, so the right interval for y is $\big[\max\{\omega^-, z^{L-} + z^{U-} - x\}, z^{L+} + z^{U+} - x\big]$.

Bringing together the two cases, we get the general expression for the appropriate interval for y, that is $\Lambda(x) = \big[\max\{\omega^-, z^{L-} + z^{U-} - x\}, z^{L+} - \max\{0, x - z^{U+}\}\big]$.

The rest of the proof is straightforward. Since by definition $\varphi^1(x)$ is the largest value of $F^A(t) - F^B(t)$ for $t \in \Lambda(x)$, we necessarily have $\overline{F}^A(x) - \overline{F}^B(x) + F^A(y) - F^B(y) \leqslant 0 \; \forall y \in \Lambda(x)$ if $\overline{F}^A(x) - \overline{F}^B(x) + \varphi^1(x) \leqslant 0$. The same line of reasoning yields Proposition 8.

B.4 Proof of Propositions 11 and 12

Let first consider the case $x \in [z^{U-}, z^{U+}]$. It can then easily be seen that the largest potential value of δ^r is $\delta^{r+} = \frac{x - z^{U-}}{\omega^+ - z^{U-}}$ (δ^r is a decreasing function of z^U) while the lowest is $\delta^{r-} = 0$. For a given set of poverty lines z^L, z^U, the value y within the "loss" poverty domain that yields the same relative gap as x is $z^L - \delta^r(z^L - \omega^-)$. Since $z^L \in [z^{L-}, z^{L+}]$, we then have $y \in [z^{L-} - \delta^r(z^{L-} - \omega^-), z^{L+} - \delta^r(z^{L+} - \omega^-)]$ for a given value of δ^r. Taking into account that $\delta^r \in \Big[0, \frac{x - z^{U-}}{\omega^+ - z^{U-}}\Big]$, we then have $y \in \Big[z^{L-} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^{L-} - \omega^-), z^{L+}\Big]$.

Now, let have a look at the case $x \in [z^{U+}, \omega^+]$. Potential values of δ are then $\delta^{r+} = \frac{x - z^{U-}}{\omega^+ - z^{U-}}$ and $\delta^{r-} = \frac{x - z^{U+}}{\omega^+ - z^{U+}}$. For a given value of x, taking the variability of z^L and z^U into account, we obtain $y \in [z^{L-} - \delta^{r+}(z^{L-} - \omega^-), z^{L+} - \delta^{r-}(z^{L+} - \omega^-)] = \Big[z^{L-} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^{L-} - \omega^-), z^{L+} - \frac{x - z^{U+}}{\omega^+ - z^{U+}}(z^{L+} - \omega^-)\Big]$.

Bringing together the two cases, we get the general expression for the appropriate interval for y, that is $\Lambda^r(x) = \Big[z^{L-} - \frac{x - z^{U-}}{\omega^+ - z^{U-}}(z^{L-} - \omega^-), z^{L+} - \max\Big\{0, \frac{x - z^{U+}}{\omega^+ - z^{U+}}(z^{L+} - \omega^-)\Big\}\Big]$.

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