Welfare gains from illiquid annuities*

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Abstract

In this article, we challenge the common thought that the annuity contract proposed by Yaari in his seminal 1965's paper is optimal. We indeed show, in a standard neo-classical framework, that another contract, which actually resembles much more to the contracts that are proposed in the "real world", may be preferred by rational individuals. According to this contract, the annuities are an illiquid asset and the premium is age independent. In an overlapping-generation economy, we show that Yaari's annuities are preferred only if the equilibrium is dynamically inefficient. Alternatively, an equilibrium displaying a positive demand for illiquid annuities is efficient. We conclude by showing how to implement an illiquid annuities market in an efficient economy.

First Draft

1 Introduction

In this article, we challenge the common thought that the annuity contract proposed by Yaari in his seminal 1965's paper is optimal. We indeed show, in a standard neo-classical framework, that another contract, which actually resembles much more to the contracts that are proposed in the "real world", may be preferred by rational individuals.

The economic theory of annuities has been strongly influenced by Yaari (1965). He studies the optimal demand for annuities in a life-cycle model with or without bequest motives. The financial asset that is named annuity

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by Yaari has the following characteristics: the returns are positive if the bearer is alive and zero if s/he is not. Annuities are nevertheless demanded since their returns are larger than the one yielded by risk-free bonds. The difference between the two returns is the annuity premium, which is said to be fair when it equals the inverse of the survival probability. Importantly, as the individual ages, the premium increases. This characterization of an annuity has been quite influential and has lead to numerous studies.

In the "real world", there are many types of annuity contracts. However, their common features are quite different from Yaari's annuities. First, the premium is age-independent. The individual purchases some annuities during youth and, at a given age -let say the age at retirement- s/he receives a fixed amount each month till death. Second, the contract is irreversible. Once payments have begun, one can not recover the amount invested. An implicit assumption in Yaari is that individuals receive the principal and the interests of their annuity after each period of investment. This means that they are in position to renegotiate their contract at each period, and that's why the premium increases as the individual ages.

In this article, we propose a standard framework in which the individual has the choice between to types of annuity contract. The first one that we named flexible, is the one proposed by Yaari (1965). The second one, which is named fixed for symmetry, is irreversible and proposes age-independent returns. In both case, we suppose that the annuity premium is such that the insurance companies make no profit. We begin our study by analyzing the life-cycle optimal decision under uncertain lifetime. Importantly, we consider a setting in which the individual ages, which more precisely means that survival probabilities decrease with age. We therefore depart from two-period life-cycle models or Blanchard (1985)'s types of model in which our distinction between flexible and fixed returns makes no sense. We obtain that fixed annuities may be preferred to flexible one if the expected returns of the first are sufficiently greater than those of the second. This is the consequence of an arbitrage between more flexibility and more returns. We then consider the general equilibrium of our economy, in which returns of both contracts are determined by the markets. We study a simple overlapping generation economy of pure exchange, similar to the one analyzed by Samuelson (1958). Surprisingly, we show that fixed annuities are preferred when the equilibrium is dynamically efficient while flexible annuities are preferred when it is inefficient. We conclude that annuity contracts proposed by Yaari (1965) are not optimal. To test the robustness of our result, we propose three extensions of our model by considering successively many periods of life, economies with capital and subjective survival probabilities.

2 Individual welfare

This section presents the annuity contracts and solves the portfolio problem in a life-cycle perspective.

2.1 Demographic structure

We consider an overlapping generations economy where agents live for a maximum of three periods, namely $0, 1$ and 2. We assume that only survivals to periods 2 and 3 are uncertain and denote by p_i , $i = 1, 2$, the probability of being alive at age i conditional on being alive at age $i - 1$. We suppose that the survival probabilities are time-independent and cannot increase with the age, i.e. $1 > p_1 > p_2 > 0$.

In each period $t = 0, 1, 2...$, N_{0t} identical individuals are born. Let us denote the number of individuals at age i at period t by N_{it}. Hence, $N_{1t+1} = p_1 N_{0t}$ and $N_{2t+2} = p_1 p_2 N_{0t}$ are the number of individuals at age 1 and 2 born at period t. The number of individuals of age 0 is suppose to grows at a constant rate, n, with $n > -1$:

$$N_{0t} = (1 + n) N_{0t-1}. \tag{1}$$

2.2 Annuity markets

There are two kinds of financial product: bonds with risk-free returns and annuities. The risk-free bond in the economy paying a real rate of return R_t in period t for one unit of consumption good invested in period $t - 1$. Due to the lifespan uncertainty, annuities markets are introduced for individuals at age 0 and 1. While the risk-free bond is available whether the consumer is alive or not, the returns of an investment in any type of annuity are only available if the consumer is alive. Two types of annuity contracts, which are described below, are offered by insurance companies. As a point of reference, we consider the case that there is perfect information about the survival probabilities and perfect competition among the insurance firms in each annuities market which ensures that insurance firms propose fair contracts. Clearly, this implies that the annuity companies make zero profits.

2.2.1 Flexible annuities contract

This contract corresponds to the actuarially-fair life annuities proposed first by Yaari (1965). This contract can be renegotiated at each period. Since insurance profit is nil, the risk premium is equal to the inverse of the survival probability. Consequently, the return of one unit of consumption good invested in period $t-1$ by an individual at age i, $i=0,1$, is simply R_t/p_{i+1} units in period t. We denote respectively by A_{0t} and A_{1t+1} the demand for flexible annuities at respectively age 0 and 1, for an individual born in t.

2.2.2 Illiquid annuities contract

This contract has two characteristics: (i) premium are independent of the age; (ii) it is irreversible: the amount invested at age 0 is necessary the same as the one invested at age 1. Loosely speaking, the capital is frozen and the individual only receives the interest factors until the last period of life: the asset is illiquid. The demand at ages 0 and 1 for illiquid annuities of an individual born in period t is denoted B_t.

Since insurer profits are nil, the risk premium at period t, denoted by R_t/π_t, verifies the equality between the expected investment returns made at $t-1$ and the expected payouts (capital and interest). The expected returns of capital invested in risk-free asset at $t-1$ in period t are:

$$\left(N_{0t-1}B_{t-1}+N_{1t-1}B_{t-2}\right)R_t, \tag{2}$$

and the expected payouts:

$$\left(N_{1t}B_{t-1}+N_{2t}B_{t-2}\right)\frac{R_t}{\pi_t}. \tag{3}$$

Consequently, the inverse of the premium solves:

$$\pi_t=\frac{p_1\left(1+n\right)B_{t-1}+p_1p_2B_{t-2}}{\left(1+n\right)B_{t-1}+p_1B_{t-2}}. \tag{4}$$

From this equation, we conclude that if the demands are non negative, the premium is greater than the one proposed by the flexible contract at age 1 and lower that the one proposed at age 2. Indeed, we have $\pi_t\in[p_2,p_1]$. The older is the individual, the more advantageous is the flexible contract. In this latter case, it is also important to remark that the annuity premium decreases with the population growth rate. Indeed, the annuity contract can be seen as an intergenerational transfer whose direction is toward the youngest.

Finally, we note that in the two limit cases: $p_1=p_2$ and $p_2=0$, the risk premium is equal to the the inverse of the survival probability at age 1, i.e. $\pi_t=p_1$. In these cases, an individual would be indifferent between contracts A_{0t} and B_t.

2.2.3 Expected returns of the two types of contract

Let us now compare the two contracts' expected returns of one unit of consumption good invested at age 0 at date t. The flexible contract expected return over the two periods is:

$$E_{A_{0,t}} = p_1 \frac{R_{t+1}}{p_1} + p_1 p_2 \left[1 + \left(\frac{R_{t+1}}{p_1} - 1 \right) \right] \frac{R_{t+2}}{p_2} = R_{t+1} + R_{t+1} R_{t+2}. \quad (5)$$

In this last equation, we have distinguished the capital, i.e. 1, and the interest return, $R_{t+1}/p_1 - 1$, in order to emphasize the assumption of an re-investment of both of them in a similar annuity contract.

The illiquid annuity contract is a little bit different in the sense that the interest returns cannot be invested in this contract and thus are invested in the flexible one. The illiquid annuity contract expected return over the two periods is:

$$E_{B_t} = p_1 \frac{R_{t+1}}{\pi_{t+1}} + p_1 p_2 \left[1 \times \frac{R_{t+2}}{\pi_{t+2}} + \left(\frac{R_{t+1}}{\pi_{t+1}} - 1 \right) \frac{R_{t+2}}{p_2} \right]. \quad (6)$$

It is clear that the expected return, E_{B_t}, increases with the premium, $1/\pi$, and is larger than $E_{A_{0,t}}$ if $\pi = p_2$ and lower $E_{A_{0,t}}$ if $\pi = p_1$. We conclude there exists a threshold $\bar{\pi}$ such that $E_{B_t} \geq E_{A_{0,t}}$ if and only if the premium is larger than $1/\bar{\pi}$. Everything else equal, we also conclude that the threshold decreases with the population growth rate.

At steady-state the analysis is easier as the inverse of the annuity premium is:

$$\pi = \frac{p_1 (1+n) + p_1 p_2}{(1+n) + p_1}, \quad (7)$$

which implies that the expected returns of the illiquid annuity contract are:

$$E_B = \frac{(1+n+p_1)}{1+n+p_2} (1 + p_2 + R) R - p_1 R. \quad (8)$$

Using expected values in (5) and (8), the comparison becomes extremely simple:

$$E_A \geq E_B \Leftrightarrow (n - R)(p_1 - p_2) \geq 0. \quad (9)$$

Provided that $p_1 > p_2$, the expected returns are the same if the interest factor is equal to the demographic growth rate. For realistic values of parameter (i.e. for $n - R > 0$), the illiquid annuity contract expected return is higher than the flexible one. We also notice that in the limit case of constant survival probabilities, $p_1 = p_2$, the two expected returns would be the same.

2.3 Optimal portfolio allocation

Individuals receive an age-dependant endowment W_i at each age i that satisfies $W_0 > 0$, $W_1 \geq 0$ and $W_2 > 0$. We suppose that investments in the illiquid annuity is positive and bounded by the endowment. This implies a short selling constraint on the demand for flexible annuities at period 0. The budget constraints of an individual born in period t in her/his first-period of life is[1]:

$$\begin{aligned} W_0 &= A_{0t} + B_t, && (10) \\ 0 &\leq B_t \leq W_0. && (11) \end{aligned}$$

Upon survival, the individual of age 1 can invest $A_{1,t+1}$ in flexible annuity contract, has to reinvest the amount chosen in the first period, B_t in illiquid annuity contract in order to make additional provision for consumption in period 2, and consumes an amount C_{1t+1}. This gives us the budget equation for the first period of retirement:

$$C_{1t+1} = W_1 + A_{0t}\frac{R_{t+1}}{p_1} + B_t\left(\frac{R_{t+1}}{\pi_{t+1}} - 1\right) - A_{1,t+1}. \quad (12)$$

Finally, the individual consumes all her/his revenue if s/he is alive in the third period of life:

$$C_{2t+2} = W_2 + A_{1t+1}\frac{R_{t+2}}{p_2} + B_t\frac{R_{t+2}}{\pi_{t+2}}. \quad (13)$$

Preferences over lifetime consumption of an individual are time-separable and represented by expected utility with a per-period utility function U depending on consumption:

$$p_1\left[U\left(C_{1t+1}\right) + p_2\theta U\left(C_{2t+2}\right)\right], \quad (14)$$

where the discount factor, θ is positive. We suppose that there is no bequest motive. The instantaneous utility function U is twice continuously differentiable, strictly increasing, concave and $U'(0) = +\infty$.

Using the Lagrangian function, the individual maximizing problem writes:

$$\begin{aligned} \max_{B_t, A_{1,t+1}} \; & U\left(W_1 + (W_0 - B_t)\frac{R_{t+1}}{p_1} + B_t\left(\frac{R_{t+1}}{\pi_{t+1}} - 1\right) - A_{1,t+1}\right) \\ & + p_2\theta U\left(W_2 + A_{1t+1}\frac{R_{t+2}}{p_2} + B_t\frac{R_{t+2}}{\pi_{t+2}}\right) + \mu_t\left(W_0 - B_t\right) + \lambda_t B_t. \quad (15) \end{aligned}$$

[1]For simplicity, we suppose that consumption at the first period, C_{0t}, is equal to zero. An alternative assumption were to consider that consumption level is given.

where μ_t and λ_t are the Lagrangian multipliers associated to the two inequality constraints imposed on B_t.

The first order conditions are:

$$-\left(\frac{R_{t+1}}{p_1}-\frac{R_{t+1}}{\pi_{t+1}}+1\right)U'\left(C_{1t+1}\right)+\frac{R_{t+2}}{\pi_{t+2}}p_2\theta U'\left(C_{2t+2}\right)+\lambda_t-\mu_t = 0 \quad (16)$$

$$-U'\left(C_{1t+1}\right)+R_{t+2}\theta U'\left(C_{2t+2}\right) = 0 \quad (17)$$

$$\mu_t\left(W_0-B_t\right) = 0 \quad (18)$$

$$\lambda_t B_t = 0 \quad (19)$$

By rewriting relations (17) and (16) that correspond to the optimal conditions on A_{1t} and B_t, we obtain:

$$-\left[\frac{R_{t+1}}{p_1}-\frac{R_{t+1}}{\pi_{t+1}}+1-\frac{p_2}{\pi_{t+2}}\right]R_{t+2}\theta U'\left(C_{2t+2}\right)+\lambda_t-\mu_t=0. \quad (20)$$

and

$$U'\left(C_{1t+1}\right)=R_{t+2}\theta U'\left(C_{2t+2}\right) \quad (21)$$

Equation (20) corresponds to the trade-off between the two types of annuities contracts. Equation (21) corresponds to the trade-off between consumptions at ages 1 and 2. This equation states that the representative agent chooses consumption such that the marginal rate of substitution between age 1 and age 2 consumptions is equal to the marginal rate of transformation. Whatever the contract chosen, the famous Yaari's result such that dynamic consumptions are independent from survival probabilities still hold.

Following Kuhn-Tucker conditions, we cannot have simultaneously $\lambda_t > 0$ and $\mu_t > 0$. Obviously, an individual always has a positive demand for annuity at age 0 but s/he can decide either to buy the flexible annuity contract (i.e. $A_{0,t} = W_0$, $B_t = 0$) or the illiquid annuity contract (i.e. $A_{0,t} = 0$, $B_t = W_0$) or both kind of contracts (i.e. $A_{0,t} > 0$, $B_t > 0$). The following proposition shows that the latter case is in general excluded.

Proposition 1 *The arbitrage condition between the two types of contracts is:*

- *If* $\frac{R_{t+1}}{p_1}-\frac{R_{t+1}}{\pi_{t+1}}+1-\frac{p_2}{\pi_{t+2}} > 0$, *then the individual only buys the flexible annuities contract at age 0;*
- *If* $\frac{R_{t+1}}{p_1}-\frac{R_{t+1}}{\pi_{t+1}}+1-\frac{p_2}{\pi_{t+2}} = 0$, *then the individual buys both types of contracts;*
- *If* $\frac{R_{t+1}}{p_1}-\frac{R_{t+1}}{\pi_{t+1}}+1-\frac{p_2}{\pi_{t+2}} < 0$, *then the individual only buys the illiquid annuities contract at age 0.*

Proof. It immediately comes from optimal conditions and relation (20). ■

To interpret the proposition, let us turn to the particular case of a steady state economy. The demand for illiquid annuities depends on the sign of the expression $\left(\frac{\pi}{p_1} - 1\right) R + \pi - p_2$, which using the value of π given by relation (7), rewrites:

$$\frac{(p_1 - p_2)(1 + n - R)}{1 + n + p_1}. \tag{22}$$

One can notice, using (9), that:

$$(p_1 - p_2)(1 + n - R) = (p_1 - p_2) + (E_A - E_B)(1 + n + p_1). \tag{23}$$

If the expected returns of the two contracts are the same, $E_A = E_B$, for example, due to the equality between interest factor and growth rate of young population, $R = n$, then the flexible annuity contract is preferred at age 0: the individuals do not invest in illiquid annuities. In the limit case such that the survival probabilities are the same, $p_1 = p_2$, we retrieve the extreme case of indifference between the two contracts at age 0.

Remark. If there were no inequality constraints on B_t, the problem would be of no interest since either the individual would sell an infinite amount of flexible annuities in period zero to buy an infinite amount of illiquid annuities, or there would be no solution (if $\frac{R_{t+1}}{p_1} - \frac{R_{t+1}}{\pi_{t+1}} + 1 \geq 0$).

Anticipating our results, at the golden rule level, $R = 1 + n$, individuals choose to buy both type of contracts. In case of over-accumulation, $R < 1 + n$, individuals only buy flexible annuity contracts. In case of under-accumulation, $R > 1 + n$, individuals only buy fixed annuity contracts at age 0. At age 1, they can also buy some flexible annuity contract. Consequently, at the optimum, the annuity contract *à la* Yaari is never optimal for individuals. Indeed, when $R = 1 + n$, flexible annuities offers a lower rate of return but more flexibility. Individual thus diversify their optimal portfolio. For a higher rate of return, the gap between the two expected returns increases and individuals are incited to give up flexible annuities. Conversely, for a lower rate of return, individuals are incited to keep flexible annuities.

The fixed contract can be viewed as a transfer from the old to the young. The individual accepts low remuneration at age 2 compensated by a higher remuneration at age 1. Consequently, the lower the population, the more attractive is the fixed contract. Equilibrium and optimality considerations will be discussed in section 3.

2.4 Discussion

Our results are robust to standard extensions.

2.4.1 Bequests

Let us first suppose that the individual has a bequest motive similar to the one studied in Yaari (1965) and let us denote by H_{it} the amount that is bequeathed if death occurs at age i and period t. The budgets constraints are modified as follows:

$$
\begin{aligned}
W_0 &= A_{0t} + B_t + H_{1,t}, && (24)\\
C_{1t+1} &= W_1 + A_{0t}\frac{R_{t+1}}{p_1} + B_t\left(\frac{R_{t+1}}{\pi_{t+1}} - 1\right) - A_{1,t+1} + H_{1,t}R_{t+1} - H_{2,t+1}, && (25)\\
C_{2t+2} &= W_2 + A_{1t+1}\frac{R_{t+2}}{p_2} + B_t\frac{R_{t+2}}{\pi_{t+2}} + H_{2,t+1}R_{t+2} - H_{3,t+2}. && (26)
\end{aligned}
$$

Preferences over lifetime consumption are modified to take into account the bequest motive. The capitalized value of the bequest then enter the expected utility.

$$
\begin{aligned}
&p_1\left[U\left(C_{1t+1}\right) + p_2\theta U\left(C_{2t+2}\right)\right] + (1-p_1)v(R_{t+1}R_{t+2}H_{1,t})\\
&+p_1(1-p_2)v(R_{t+2}H_{2,t+1}) + p_1p_2v(H_{3,t+2}) \qquad (27)
\end{aligned}
$$

We suppose that the utility function v is twice continuously differentiable, strictly increasing. The optimal conditions are similar to equations (17) and (16) for respectively A_{0t} and B_t and for bequests:

$$
\begin{aligned}
-U'\left(C_{1t+1}\right) + R_{t+2}v'(R_{t+1}R_{t+2}H_{1,t}) &= 0 \quad (28)\\
-U'\left(C_{1t+1}\right) + p_2R_{t+2}\theta U'\left(C_{2t+2}\right) + (1-p_2)R_{t+2}v'(R_{t+2}H_{2,t+1}) &= 0 \quad (29)\\
-\theta U'\left(C_{2t+2}\right) + v'(H_{3,t+2}) &= 0 \quad (30)
\end{aligned}
$$

By rewriting relations (28), (29) and (30), we deduce:

$$
-v'(R_{t+1}R_{t+2}H_{1,t}) + (1-p_2)R_{t+2}v'(R_{t+2}H_{2,t+1}) + p_2v'(H_{3,t+2}) = 0. \qquad (31)
$$

Consequently, the optimal bequests are such that:

$$
H_{3,t+2} = R_{t+2}H_{2,t+1} = R_{t+1}R_{t+2}H_{1,t}. \qquad (32)
$$

The second lesson from Yaari still holds: the capitalized values of bequest are independent from the age of death.

2.4.2 Subjective probabilities

The way people perceive risks is often subjective and intuitive and they frequently view risks in a way that differs from scientific assessments. Many studies have demonstrated the importance of probability distortion in risky choice (see among others Tversky and Kahneman, 1992; Camerer and Ho, 1994; Wu and Gonzalez, 1996; Gonzalez and Wu, 1999; Abdellaoui, 2000). As Brewer *et al.* (2007) showed the consistent relationships between risk perceptions and behavior suggest that risk perceptions are rightly placed as core concepts when individuals face a health risk and thus a longevity one. Since thirty years, different models of representation of preferences under uncertainty are proposed. Moreover, in case of two states of nature, the main models reach to one: the subjective model introduced by Savage (1954).

Let us consider an individual endowed with the subjective survival probability $\hat{p}_2$. Her/his preferences are represented by a subjective expected utility:

$$U\left(C_{1t+1}\right)+\hat{p}_2\theta U\left(C_{2t+2}\right),$$

with the usual assumptions on the utility function U. The two first optimal conditions become:

$$\begin{aligned}-\left(\frac{R_{t+1}}{p_1}-\frac{R_{t+1}}{\pi_{t+1}}+1\right)U'\left(C_{1t+1}\right)+\frac{R_{t+2}}{\pi_{t+2}}\hat{p}_2\theta U'\left(C_{2t+2}\right)+\lambda_t-\mu_t &= 0\\ -U'\left(C_{1t+1}\right)+\frac{\hat{p}_2}{p_2}R_{t+2}\theta U'\left(C_{2t+2}\right) &= 0\end{aligned}$$

And their combination:

$$-\left(\frac{R_{t+1}}{p_1}-\frac{R_{t+1}}{\pi_{t+1}}+1-\frac{p_2}{\pi_{t+2}}\right)\frac{\hat{p}_2}{p_2}R_{t+2}\theta U'\left(C_{2t+2}\right)+\lambda_t-\mu_t=0$$

Consequently, the optimal rule is the same as previously. This result is particularly relevant when individuals' beliefs differ from the insurers' survival probabilities estimation. A natural question which arises concerns the possibility for insurers to use this information and modify payouts.

3 Equilibrium and optimality

This section studies the intertemporal equilibrium of an endowment economy and discusses the Pareto-optimality of the market allocation. The extension to an economy with productive capital is then discussed.

3.1 Intertemporal equilibrium in a pure exchange economy

In this subsection, the intertemporal equilibrium is studied in a simple endowment economy similar to the one proposed by Samuelson (1958). This is an overlapping generations model of pure exchange with one type of individual and one commodity. It is well-known since Gale (1973) that this kind of economy has two types of equilibria: balanced ones, with zero net savings; and the golden rule, in which the aggregate savings can be positive. We focus here on the first type and study the existence of steady-states. Positive aggregate savings could also be considered by introducing money, which would be an immediate extension of what will be developed below, or physical capital as suggested in a following subsection.

There is one goods market on which is traded non-storable goods whose aggregate endowment is given and whose distribution among cohorts is stationary. The equilibrium condition at time t is given by the equality between aggregate endowments and aggregate consumption, such that:

$$N_{0t}W_0 + N_{1t}W_1 + N_{2t}W_2 = N_{1t}C_{1t} + N_{2t}C_{2t}, \tag{33}$$

which can be rewritten as follows:

$$W_0 + \frac{p_1}{1+n}W_1 + \frac{p_1 p_2}{(1+n)^2}W_2 = \frac{p_1}{1+n}C_{1t} + \frac{p_1 p_2}{(1+n)^2}C_{2t}. \tag{34}$$

On the asset market, consumption loans are available at the risk-free interest rate R_t. The two kinds of annuity contracts we consider are also proposed to ensure the individual against the uncertainty on longevity. The equilibrium on the asset market at time t imposes that aggregate savings equals zero, which is given by the following condition:

$$W_0 + \frac{p_1}{1+n}\left(B_{t-1} + A_{1t}\right) = 0. \tag{35}$$

Hence, a balanced equilibria necessarily implies that individuals sell annuities short at age 1. It is easy to check that condition (35) implies (34), but that the reverse is not always true. Let us now study the stationary solutions.

Property. *The stationary solution of the equilibrium is a* (B, R, λ, μ) *solution of the following system:*

$$-\frac{(p_1 - p_2)(1+n-R)}{1+n+p_1}\theta R\frac{p_2}{p_1}U'(C_2) + \lambda - \mu = 0, \tag{36}$$

$$-U'(C_1) + R\theta U'(C_2) = 0, \tag{37}$$

$$\mu(W_0 - B) = 0, \tag{38}$$

$$\lambda B = 0, \tag{39}$$

with:

$$C_1 = W_1 + W_0 \frac{R+1+n}{p_1} + B \frac{R(p_1 - p_2)}{p_1(1+n+p_2)}, \quad (40)$$

$$C_2 = W_2 - \frac{(1+n)R}{p_1 p_2} W_0 - B \frac{R(p_1 - p_2)(1+n)}{p_1 p_2 (1+n+p_2)}. \quad (41)$$

Proof. At steady-state, the equilibrium condition (35) writes $B + A_1 = -\frac{1+n}{p_1} W_0$ and the no-profit condition (7) writes $\pi = \frac{p_1(1+n)+p_1 p_2}{1+n+p_1}$. This implies that individual consumptions can be expressed as functions of B, as done in (40) and (41). Finally, equations (36), (37), (38), and (39) are derived from the individual's problem using the no-profit condition (7). ■

Proposition 2. *There exists a unique steady-state. Depending on parameter's values, the steady-state satisfies $R < 1+n$, $R = 1+n$, or $R > 1+n$.*

Proof. We are going to show that there exists a unique equilibrium, that satisfies $R < 1+n$ if $x_1 > 0$, $R = 1+n$ if $x_1 < 0 < x_2$ and $R > 1+n$ if $x_2 < 0$, where the values of x_1 and x_2 are such that:

$$x_1 = W_2 - \frac{1+n+p_1}{1+n+p_2} \frac{(1+n)^2}{p_1 p_2} W_0$$

$$x_2 = W_2 - \frac{(1+n)^2 W_0}{p_1 p_2}$$

We proceed in two steps. Step 1: suppose that $R \neq 1+n$. From Proposition 1, we know that $B = 0$ if $R < 1+n$, and that $B = W_0$ if $R > 1+n$. The problem is hence to find the real root of $\phi(R) = 0$ where function ϕ is picewise-defined as follows:

$$\phi(R) = -U'\left(W_1 + \frac{\varepsilon R + 1 + n}{p_1} W_0\right) + R\theta U'\left(W_2 - \frac{\varepsilon(1+n)}{p_1 p_2} W_0 R\right),$$

where

$$\varepsilon = \begin{vmatrix} 1 & \text{if } R < 1+n, \\ \frac{1+n+p_1}{1+n+p_2} & \text{if } R > 1+n. \end{vmatrix}$$

We have $\phi'(R) > 0$, $\phi(0) < 0$ and $\phi\left(p_1 p_2 W_2 / \left[(1+n) \frac{(1+n+p_1)}{(1+n+p_2)} W_0\right]\right) > 0$. Then, we conclude there exists a unique R that satisfies $\phi(R) = 0$ and $R < 1+n$ if $\lim_{R \to (1+n)^-} \phi(R) > 0$. Similarly, there exists a unique R that satisfies $\phi(R) = 0$ and $R > 1+n$ if $\lim_{R \to (1+n)^+} \phi(R) < 0$. Finally, there is no R that satisfies $\phi(R) = 0$ if $\lim_{R \to (1+n)^-} \phi(R) < 0 < \lim_{R \to (1+n)^+} \phi(R)$.

Step 2: suppose that $R=1+n$. From Proposition 1, we know that $\lambda=\mu=0$. The problem is now to find whether the application $\psi(B)=0$ has a real root that belongs to $(0,W_1)$, with ψ defined by:

$$\begin{aligned}\psi(B) &= -U'\left(W_1+\frac{2(1+n)}{p_1}W_0+B\frac{(1+2)(p_1-p_2)}{p_1(1+n+p_2)}\right)\\ &\quad +(1+n)\theta U'\left(W_2-\frac{(1+n)^2}{p_1p_2}W_0-B\frac{(p_1-p_2)(1+n)^2}{p_1p_2(1+n+p_2)}\right).\end{aligned}$$

Since $\psi'(B)>0$, the condition for the existence of the existence of a unique root is $\psi(0)<0<\psi(W_0)$.
To conclude, let us notice that $\psi(0)=\lim_{R\to(1+n)^-}\phi(R)$ and let us name this limit x_1:

$$x_1=-U'\left(W_1+\frac{2(1+n)}{p_1}W_0\right)+R\theta U'\left(W_2-\frac{(1+n)^2}{p_1p_2}W_0\right),$$

which is a real number if: $W_2-\frac{(1+n)^2}{p_1p_2}W_0>0$. Similarly, we have $\psi(W_0)=\lim_{R\to(1+n)^+}\phi(R)$, and define x_2 such that:

$$x_2=-U'\left(W_1+\frac{\left(\frac{1+n+p_1}{1+n+p_2}+1\right)(1+n)}{p_1}W_0\right)+R\theta U'\left(W_2-\frac{\frac{1+n+p_1}{1+n+p_2}(1+n)^2}{p_1p_2}W_0\right)$$

which is a real number if: $W_2-\frac{1+n+p_1}{1+n+p_2}\frac{(1+n)^2W_0}{p_1p_2}>0$. It is simple to show that $x_1<x_2$, provided they are real numbers. The decomposition we proposed in the beginning of the proof is then immediate. ■

With this simple endowment economy, we are hence able to claim that there are sets of parameters such that holding illiquid annuities is an equilibrium output. Let us now turn to the stability property of the stationary solutions.

Since time does not extend from an infinite past, we have to distinguish the individuals who are born at date $t=0,1,...$ from those who were born before and who are still alive at date $t=0$. Individuals born at date $t=0,1,...$ solve the problem (15) and their first order conditions are given by (16), (17), (18) and (19). A sequence of pairs $(B_t,A_{1,t+1})$ for $t=0,1,...$ is then defined for a expected sequence of "prices" (R_t,π_t) given for $t=1,2,...$ The individual born at date $t=-1$ solves at time $t=0$ the following problem:

$$\max_{A_{1,0}} U\left(W_1+(W_0-B_{-1})\frac{R_0}{p_1}+B_{-1}\left(\frac{R_0}{\pi_0}-1\right)-A_{1,0}\right)$$
$$+p_2\theta U\left(W_2+A_{1,0}\frac{R_1}{p_2}+B_{-1}\frac{R_1}{\pi_1}\right) \tag{42}$$

with $B_{-1}\in[0,W_0]$ given. The "prices" that are relevant for this individual are thus (R_0,R_1,π_0,π_1). By the definition of π_t, given in (4), we conclude that π_0 is given. Finally, the individual born at date $t=-2$ does not make any choice and simply consumes his/her endowment.

The equilibrium is thus the solution of the following system:

$$\begin{cases} -\left(\frac{R_{t+1}}{p_1}-\frac{R_{t+1}}{\pi_{t+1}}+1\right)U'\left(C_{1t+1}\right)+\frac{R_{t+2}}{\pi_{t+2}}p_2\theta U'\left(C_{2t+2}\right)+\lambda_t-\mu_t=0 \text{ for } t=0,1,... \\ -U'\left(C_{1t+1}\right)+R_{t+2}\theta U'\left(C_{2t+2}\right)=0 \text{ for } t=-1,0,1,... \\ \mu_t\left(W_0-B_t\right)=0 \text{ for } t=0,1,... \\ \lambda_t B_t=0 \text{ for } t=0,1,... \end{cases} \tag{43}$$

where the consumptions are obtained by replacing (35) in (12) and (13) to get:

$$\begin{cases} C_{1t+1}=W_1+W_0\frac{1+n}{p_1}+\left(W_0-B_t\right)\frac{R_{t+1}}{p_1}+B_t\frac{R_{t+1}}{\pi_{t+1}}, \\ C_{2t+2}=W_2-W_0\frac{1+n}{p_1}\frac{R_{t+2}}{p_2}+B_t\left(\frac{R_{t+2}}{\pi_{t+2}}-\frac{R_{t+2}}{p_2}\right), \end{cases} \tag{44}$$

and with $\pi_0>0$ and B_{-1} given.

Proposition 3. *Suppose that $B_{-1}=0$ and let R^* be the solution of:*

$$-U'\left(W_1+W_0\frac{1+n+R^*}{p_1}\right)+R^*\theta U'\left(W_2-W_0\frac{1+n}{p_1}\frac{R^*}{p_2}\right)=0. \tag{45}$$

Then $R_t=R^*$ for all $t=0,1,...$ Moreover, if $R^*\geq p_2$, $B_t=0$ for all $t=0,1,...$ If $R^*<p_2$, the solution is indeterminate in B_t: $B_t=0$ for all $t=0,1,...$ is a solution and there exists $\hat{B}\in(0,W_0)$ such that any sequence that satisfies, for any date $t=\hat{t}$, $B_{\hat{t}}=\hat{B}$ and $B_{\hat{t}+1}=0$ is also a solution.

Proof. We proceed in four steps. We first show that there exists only two kind of possible sequences: the first one satisfies $B_t=0$ for all $t=0,1,...$ and the second one admits interior solutions at a given date $\hat{t}$: $B_{\hat{t}}\in(0,W_0)$ provided that $B_{\hat{t}+1}=0$ for all $t=0,1,....$ Second, we show that the first

sequence is always a solution while the ones of the second kind are a solution only if $R_{\hat{t}+2} < \left(1 - \frac{p_2}{p_1}\right) / \left(\frac{1}{p_2} - \frac{1}{p_1}\right) = p_2$. Third, we show that $R_t = R^*$ for all $t = 0, 1, ...,$ where R^* is the solution of (45). Finally, we show that there exists sets of parameters such that $R^* < p_2$.
Step 1. For $B_{-1} = 0$, one has, using (4), $\pi_1 = p_1$. By replacing this equality in the conditions given in Proposition 1, we obtain that $B_0 = 0$ if $1 - p_2/\pi_2 > 0$, which turn to be true since for $B_0 = 0$, one has $\pi_2 = p_1$. We conclude that the sequence $B_t = 0$ for all $t = 0, 1, ...$ is a possible solution. However, from Proposition 1, we also have that $B_0 \in (0, W_0)$ if $1 - p_2/\pi_2 = 0$, which can be obtained if $\pi_2 = p_2$. Using (4), we hence conclude that $B_0 \in (0, W_0)$ can be a solution if and only if $B_1 = 0$, which is obtained (using again Proposition 1) if $R_2 < \left(1 - \frac{p_2}{p_1}\right) / \left(\frac{1}{p_2} - \frac{1}{p_1}\right)$. This implies that $\pi_3 = p_1$. Then, the same reasoning applies and B_2 can either be equal to 0 or belong to $(0, W_0)$. To summarize: $B_t = 0$ for all t is a solution, $B_{2t} \in (0, W_0)$ and $B_{2t+1} = 0$ is also a solution, and all possibilities between those two cases are also solutions.
Step 2. It follows from step 1, that $B_{\hat{t}} \in (0, W_0)$ is possible if only if $\pi_{\hat{t}+1} = p_1$, $\pi_{\hat{t}+2} = p_2$ and $R_{\hat{t}+2} < \left(1 - \frac{p_2}{p_1}\right) / \left(\frac{1}{p_2} - \frac{1}{p_1}\right)$.
Step 3. Let us denote by $\left(C_{1,\hat{t}+1}, C_{2,\hat{t}+2}\right)$ the consumption bundle of an individual who chooses $B_{\hat{t}} \in (0, W_0)$ and by $(C_{1,t+1}, C_{2,t+2})$ the consumption bundle of an individual who chooses $B_t = 0$. We notice that:

$$\begin{aligned}
C_{1,\hat{t}+1} &= C_{1,t+1} = W_1 + W_0 \frac{1+n}{p_1} + W_0 \frac{R_{t+1}}{p_1}, \\
C_{2,\hat{t}+2} &= C_{2t+2} = W_2 - W_0 \frac{1+n}{p_1} \frac{R_{t+2}}{p_2},
\end{aligned}$$

and conclude that the dynamics of R_t is given by:

$$-U' \left(W_1 + W_0 \frac{1+n+R_{t+1}}{p_1} \right) + R_{t+2} \theta U' \left(W_2 - W_0 \frac{1+n}{p_1} \frac{R_{t+2}}{p_2} \right) = 0$$

for $t = -1, 0, 1, ...$ The steady state is the real solution of (45). It is immediate that R^* exists and is unique. Since :

$$\frac{dR_{t+2}}{dR_{t+1}} = - \frac{-U''\left(C_{1t+1}\right) \frac{W_0}{p_1}}{\theta U'\left(C_{2t+2}\right) - W_0 \frac{1+n}{p_1 p_2} R_{t+2} \theta U''\left(C_{2t+2}\right)} < 0,$$

we conclude that the steady-state is unstable: $R_t = R^*$ for all $t = 0, 1, ...$
Step 4. We now show that it is possible that $R^* < \left(1 - \frac{p_2}{p_1}\right) / \left(\frac{1}{p_2} - \frac{1}{p_1}\right)$. Let us denote $R^* = \varepsilon \left(1 - \frac{p_2}{p_1}\right) / \left(\frac{1}{p_2} - \frac{1}{p_1}\right)$, we are thus willing to give a condition

such that there exist $\varepsilon \in (0,1)$ that is the solution to

$$-U'\left(W_1 + W_0\frac{1+n+\varepsilon\frac{\left(1-\frac{p_2}{p_1}\right)}{\left(\frac{1}{p_2}-\frac{1}{p_1}\right)}}{p_1}\right)+\varepsilon\frac{\left(1-\frac{p_2}{p_1}\right)}{\left(\frac{1}{p_2}-\frac{1}{p_1}\right)}\theta U'\left(W_2 - W_0\frac{1+n}{p_1}\varepsilon\right) = 0.$$

The LHS is monotonically increasing with ε, is negative for $\varepsilon = 0$ and is positive for $\varepsilon = 1$ if and only if

$$-U'\left(W_1 + W_0\frac{1+n+\frac{\left(1-\frac{p_2}{p_1}\right)}{\left(\frac{1}{p_2}-\frac{1}{p_1}\right)}}{p_1}\right)+p_2\theta U'\left(W_2 - W_0\frac{1+n}{p_1}\right) > 0,$$

which is a necessary condition on parameters for having indeterminacy. ■

Proposition 3 says that if there is no illiquid annuities at the initial date of the economy, an optimal portfolio will never be, at the equilibrium, composed of illiquid annuities only. It is very likely that individual will never hold any illiquid annuities, at least if the interest rate is positive (i.e. for $R > 1$). Indeterminacy may arise in this equilibrium, but it is not a meaningful one. It simply means that if the illiquid annuity premium is for two consecutive dates $1/p_1$ and $1/p_2$, the individual is indifferent between the two types of annuities. Those annuities however does not imply an intergenerational transfer.

Proposition 4. *Suppose that $B_{-1} \in (0, W_0]$. A trajectory that satisfies $\lambda_t = \mu_t = 0$ for all $t = 0, 1, ..,$ converges to the steady state $R = 1 + n$, which is locally indeterminate.*

Proof. We first show that if $\lambda_t = \mu_t = 0$ for all $t = 0, 1, ..,$ the system rewrites as a system of dimension 2, whose steady-state is given by $R = 1 + n$, and which is locally indeterminate.

Let us suppose that $\lambda_t = \mu_t = 0$, using (43), the dynamics are given by the following two equations:

$$-\left(\frac{R_{t+1}}{p_1} - \frac{R_{t+1}}{\pi_{t+1}} + 1\right)U'\left(C_{1t+1}\right) + \frac{R_{t+2}}{\pi_{t+2}}p_2\theta U'\left(C_{2t+2}\right) = 0,$$

$$-U'\left(C_{1t+1}\right) + R_{t+2}\theta U'\left(C_{2t+2}\right) = 0,$$

and (44). By combining those equations, we obtain that: $R_{t+1} = \left(1 - p_2/\pi_{t+2}\right) / \left(1/\pi_{t+1} - 1/p_1\right)$. Using (4), we thus obtain a three-dimensional system in (π_t, B_t, R_t) that writes:

$$\begin{cases} \pi_{t+2} = \frac{p_2}{\left(1-\left(\frac{1}{\pi_{t+1}}-\frac{1}{p_1}\right)R_{t+1}\right)}, \\ B_t = \frac{(\pi_{t+1}-p_2)}{(1+n)\left(1-\frac{\pi_{t+1}}{p_1}\right)}B_{t-1}, \\ -U'\left(C_{1t+1}\right) + R_{t+2}\theta U'\left(C_{2t+2}\right) = 0, \end{cases} \tag{46}$$

with:

$$C_{1t+1} = W_1 + W_0\frac{1+n}{p_1} + (W_0 - B_t)\frac{R_{t+1}}{p_1} + B_t\frac{R_{t+1}}{\pi_{t+1}},$$

and:

$$C_{2t+2} = W_2 - W_0\frac{1+n}{p_1}\frac{R_{t+2}}{p_2} + B_t\left(\frac{R_{t+2}}{\pi_{t+2}} - \frac{R_{t+2}}{p_2}\right).$$

We use the two last equations to express B_t and C_{2t+2} as functions of $(C_{1t+1}, R_{t+1}, \pi_{t+1}, R_{t+2}, \pi_{t+2})$, such that:

$$B_t = \frac{C_{1t+1} - W_1 - W_0\left(\frac{1+n}{p_1} + \frac{R_{t+1}}{p_1}\right)}{\left(\frac{R_{t+1}}{\pi_{t+1}} - \frac{R_{t+1}}{p_1}\right)}, \tag{47}$$

$$C_{2t+2} = W_2 - W_0\frac{1+n}{p_1}\frac{R_{t+2}}{p_2} + \frac{C_{1t+1} - W_1 - W_0\left(\frac{1+n}{p_1} + \frac{R_{t+1}}{p_1}\right)}{\left(\frac{R_{t+1}}{\pi_{t+1}} - \frac{R_{t+1}}{p_1}\right)}\left(\frac{R_{t+2}}{\pi_{t+2}} - \frac{R_{t+2}}{p_2}\right). \tag{48}$$

We replace (47) in (46) to obtain:

$$C_{1t+1} - W_1 - W_0\left(\frac{1+n}{p_1} + \frac{R_{t+1}}{p_1}\right) = \frac{R_{t+1}\left(1 - \frac{p_2}{\pi_{t+1}}\right)}{(1+n)}\frac{C_{1t} - W_1 - W_0\left(\frac{1+n}{p_1} + \frac{R_t}{p_1}\right)}{\left(\frac{R_t}{\pi_t} - \frac{R_t}{p_1}\right)}.$$

Then using the fact that $p_2/\pi_{t+1} = \left(1 - \left(1/\pi_t - 1/p_1\right)R_t\right)$, we finally have:

$$C_{1t+1} = \frac{R_{t+1}}{(1+n)}C_{1t} + \left(1 - \frac{R_{t+1}}{(1+n)}\right)W_1 + W_0\left(\frac{1+n}{p_1} - \frac{R_t}{p_1}\frac{R_{t+1}}{(1+n)}\right).$$

Moreover, by replacing $\pi_{t+2} = p_2/\left(1 - \left(1/\pi_{t+1} - 1/p_1\right)R_{t+1}\right)$ in (48), we have:

$$C_{2t+2} = W_2 - C_{1t+1}\frac{R_{t+2}}{p_2} + W_1\frac{R_{t+2}}{p_2} + W_0\frac{R_{t+1}}{p_1}\frac{R_{t+2}}{p_2}.$$

We finally obtain a two-dimensional system that writes:

$$\begin{cases} C_{1t+2} = \frac{R_{t+2}}{(1+n)}C_{1t+1} + \left(1 - \frac{R_{t+2}}{(1+n)}\right)W_1 + W_0\left(\frac{1+n}{p_1} - \frac{R_{t+1}}{p_1}\frac{R_{t+2}}{(1+n)}\right), \\ -U'\left(C_{1t+1}\right) + R_{t+2}\theta U'\left(W_2 - C_{1t+1}\frac{R_{t+2}}{p_2} + W_1\frac{R_{t+2}}{p_2} + W_0\frac{R_{t+1}}{p_1}\frac{R_{t+2}}{p_2}\right) = 0; \end{cases}$$

The second equation is an implicit equation that can be written as: $R_{t+2} = f\left(R_{t+1}, C_{1t+1}\right)$ where function f satisfies the following partial derivatives:

$$
\begin{aligned}
f'_R &= -\frac{R_{t+2}\theta U''\left(C_{2t+2}\right)\frac{W_0}{p_1}\frac{R_{t+2}}{p_2}}{\theta U'\left(C_{2t+2}\right) - \frac{R_{t+2}\theta}{p_2}U''\left(C_{2t+2}\right)\left[C_{1t+1} - W_1 - W_0\frac{R_{t+1}}{p_1}\right]} \\
f'_{C_1} &= -\frac{-U''\left(C_{1t+1}\right) - \frac{R_{t+2}}{p_2}R_{t+2}\theta U''\left(C_{2t+2}\right)}{\theta U'\left(C_{2t+2}\right) - \frac{R_{t+2}\theta}{p_2}U''\left(C_{2t+2}\right)\left[C_{1t+1} - W_1 - W_0\frac{R_{t+1}}{p_1}\right]}
\end{aligned}
$$

Since: $C_{1t+1} = W_1 + W_0\frac{1+n}{p_1} + \left(W_0 - B_t\right)\frac{R_{t+1}}{p_1} + B_t\frac{R_{t+1}}{\pi_{t+1}}$, we have $C_{1t+1} - W_1 - W_0\frac{R_{t+1}}{p_1} = W_0\frac{1+n}{p_1} + B_t\left(\frac{R_{t+1}}{\pi_{t+1}} - \frac{R_{t+1}}{p_1}\right) > 0$. Consequently: $f'_R > 0$ et $f'_{C_1} < 0$. To summarize this part, we conclude that the dynamics are given by a system of two equations involving forward variables only:

$$
\begin{aligned}
& C_{1t+2} = \frac{f(R_{t+1},C_{1t+1})}{(1+n)}C_{1t+1} + \left(1 - \frac{f(R_{t+1},C_{1t+1})}{(1+n)}\right)W_1 \\
& +W_0\left(\frac{1+n}{p_1} - \frac{R_{t+1}}{p_1}\frac{f(R_{t+1},C_{1t+1})}{(1+n)}\right) \\
& R_{t+2} = f\left(R_{t+1}, C_{1t+1}\right)
\end{aligned}
\tag{49}
$$

Let us suppose now that we are in the neigborhood of the steady-state $R = 1 + n$. By definition $\lambda_t = \mu_t = 0$ and the dynamics are given by (49). The Jacobian matrix writes:

$$
\begin{matrix}
f'_{C_1}\left[\frac{C_{1t+1}}{(1+n)} - \frac{W_1}{(1+n)} - \frac{W_0}{p_1}\right] + 1 & f'_R\left[\frac{C_{1t+1}}{(1+n)} - \frac{W_1}{(1+n)} - \frac{W_0}{p_1}\right] - \frac{W_0}{p_1} \\
f'_{C_1} & f'_R
\end{matrix}
$$

The determinant is:

$$
D = f'_R + f'_{C_1}\frac{W_0}{p_1} = \frac{\frac{W_0}{p_1}U''\left(C_1\right)}{\theta U'\left(C_2\right) - \frac{(1+n)\theta}{p_2}U''\left(C_2\right)\left[C_1 - W_1 - W_0\frac{(1+n)}{p_1}\right]} < 0
$$

There is therefore two real eigenvalues. The trace is:

$$
T = \frac{U''\left(C_1\right)\left[\frac{C_1}{(1+n)} - \frac{W_1}{(1+n)} - \frac{W_0}{p_1}\right] + \theta U'\left(C_2\right) - \left(1+n\right)\theta U''\left(C_2\right)\frac{W_0}{p_1}\frac{(1+n)}{p_2}}{\theta U'\left(C_2\right) - \frac{(1+n)\theta}{p_2}U''\left(C_2\right)\left[C_1 - W_1 - W_0\frac{(1+n)}{p_1}\right]}.
$$

We moreover compute:

$$
1 + D - T = \frac{-\frac{1}{1+n}\left[U''\left(C_1\right) + \frac{\theta(1+n)^2}{p_2}U''\left(C_2\right)\right]\left[C_1 - W_1 - 2\frac{W_0(1+n)}{p_1}\right]}{\theta U'\left(C_2\right) - \frac{(1+n)\theta}{p_2}U''\left(C_2\right)\left[C_1 - W_1 - W_0\frac{(1+n)}{p_1}\right]}
$$

Since

$$C_1 - W_1 - 2\frac{W_0\left(1+n\right)}{p_1} = -B\frac{\left(1+n\right)}{p_1} + B\frac{\left(1+n\right)}{\pi} > 0$$

we conclude that: $1 + D - T > 0$. We deduce there is at least one eigenvalue of modulus lower than one and that the steady state is locally indeterminate. ■

An important implication of Proposition 4 is that the interest factor R_t cannot converge to another steady-state than $1 + n$. The stability analysis of the other steady-states is build on this result.

Proposition 5. *Suppose that $B_{-1} \in (0, W_0]$. If the steady-state satisfies $R \in [p_2, 1+n)$, there exists $T > -1$ such that $R_{T+1+i} = R$ and $B_{T+i} = 0$ for all $i = 0, 1, ...$ If the steady-state satisfies $R > 1 + n$, $R_{t+1} = R$ and $B_t = W_0$ for all $i = 0, 1, ...$*

Proof. Let us study first the dynamics driving to steady-state such that $R < 1 + n$ and $B = 0$. This steady-state is not a solution of the interior dynamics given by (46), which implies that there is no sequence of B_t that converge to the steady-state $B = 0$. The latter is at best reached in finite time. Let us define date $T > -1$ such that $B_T = 0$. We can hence directly apply Proposition 3 to state that, for all $i = 1, 2, ...$, $R_{T+i} = R^*$ and $B_{T+i} = 0$ (provided that $R \geq p_2$).

Let us now study the dynamics driving to the steady-state such that $R > 1 + n$ and $B = W_0$. As for the previous case, this steady-state is not a solution of the interior dynamics given by (46), which implies that steady-state $B = W_0$ is at best reached in finite time. Let us consider a neighborhood of the steady-state such that $B = W_0$, This implies that $\pi = p_1\left(1+n+p_2\right)/\left(1+n+p_1\right)$, and thus:

$$\begin{aligned} C_{1t+1} &= W_1 + W_0\frac{1+n}{p_1} + W_0 R_{t+1}\frac{1+n+p_1}{p_1\left(1+n+p_2\right)}, \\ C_{2t+2} &= W_2 - W_0 R_{t+2}\frac{\left(n+p_2\right)\left(1+n+p_1\right)}{p_1 p_2\left(1+n+p_2\right)}. \end{aligned}$$

The local dynamics of R_t is thus given by:

$$-U'\left(C_{1t+1}\right) + R_{t+2}\theta U'\left(C_{2t+2}\right) = 0,$$

for which there exists a unique steady-state R^{**} that solves:

$$-U'\left(W_1 + W_0\frac{1+n}{p_1} + W_0 R\frac{1+n+p_1}{p_1\left(1+n+p_2\right)}\right) + R\theta U'\left(W_2 - W_0 R\frac{\left(n+p_2\right)\left(1+n+p_1\right)}{p_1 p_2\left(1+n+p_2\right)}\right) = 0$$

Since:

$$\frac{dR_{t+2}}{dR_{t+1}}=-\frac{-U''\left(C_{1t+1}\right)W_0\frac{1+n+p_1}{p_1(1+n+p_2)}}{\theta U'\left(C_{2t+2}\right)-R_{t+2}\theta U''\left(C_{2t+2}\right)W_0\frac{(n+p_2)(1+n+p_1)}{p_1p_2(1+n+p_2)}}<0$$

The steady-state is unstable. Consequently, $R_t = R^{**}$ for all $t = 1, 2...$ while R_0 is obtained by solving the program of the individual born at date $t = -1$ and using the equilibrium condition (35). ■

3.2 Optimality

In this subsection, we discuss the Pareto optimality of the intertemporal equilibrium defined above. We first show that the equilibrium with flexible annuities is not efficient while the one with illiquid annuities is efficient. We then discuss the issue of the implementation of the illiquid annuity contract.

Let us begin with two results on the efficiency of trajectories described in Proposition 5.

Proposition 6. *The individuals only hold flexible annuities if the economy is dynamically inefficient. Otherwise s/he holds illiquid annuities.*

Proof. Proposition 5 showed that the only trajectory that displays some $B_t = 0$ converges to a steady-state such that $R < 1 + n$. It is well-known that for such an equilibrium a simple intergenerational transfer from the young to the old constitutes a Pareto improvement. ■

Proposition 7. *If the individual hold illiquid annuities only, the economy is dynamically efficient.*

Proof. We consider the equilibrium such that $B_t = W_0$ and $R_t = R^{**} > 1+n$. A transfer from individuals of age 1 to individuals of age 2 can be seen as forced savings that returns $(1 + n)/p_1$; since annuities return:

$$\frac{R}{\pi}=\frac{R\left(1+n+p_1\right)}{p_1\left(1+n+p_2\right)},$$

such a transfer cannot constitutes a Pareto improvement. Similarly, a transfer from individuals of age 2 to individuals of age 3 can be seen as forced savings that returns $(1 + n)/p_2$, which also cannot be a Pareto improvement. Finally, the same argument applies to a transfer from individuals of age 1 to individuals of age 3. ■

Propositions 6 and 7 apply the standard argument about Pareto optimality in OLG models to our framework. We conclude that if illiquid annuities were chosen by the individuals, the equilibrium would be efficient. However, there exist a specific issue concerning the implementation of the illiquid annuity contract. Suppose that before time $t = 0$, the illiquid annuity contract did not existed. Proposition 3 says that this contract would never be implemented although it may constitute a welfare improvement for all generations born at time $t = 0, 1, ..$ provided that the long run interest rate is greater than the demographic grow rate. Actually, it is the generation born at time $t = -1$ who has no interest in choosing an annuity that return R_0/π_0 rather than the one that return R_0/p_2. In what follows we propose a debt scheme that permit to compensate the generation born at date $t = -1$ in order to implement the illiquid annuity contract.

The idea is to issue a debt at time $t = 1$ in order to make the return of illiquid annuities proposed to individuals born at date $t = -1$, equal to the return of flexible annuities, i.e. equal to R_1/p_2. This debt is reimbursed by taxing the returns of illiquid annuities proposed to all individuals born at date $t = 0, 1, ...$ Proposition 1 is thus slightly modified in order to take into account this new scheme. It is easy to show there exists $\varepsilon > 0$ such that illiquid annuities are preferred by individual if $R > 1 + n + \varepsilon$. We conclude that if the long run interest rate is sufficiently large and if there is no illiquid annuities at the initial date of the economy, issuing a debt can constitute a Pareto improvement.

3.3 An economy with capital

In this subsection, we argue that our results obtained in the exchange economy framework extend to an economy with capital.

There is one good which is either consumed or invested for future capital. The good produced at each period t is taken as the numeraire. We introduce a production technology which is the same for all periods. It is represented by a neo-classical production function, $F(K_t, L_t)$, for period t. Function F is homogeneous of degree one increasing and strictly concave with respect to its arguments, the capital, K and labor L. We suppose that the capital stock totally depreciates in one period. The initial stock of capital, K_0 is already installed in the firm producing at $t = 0$. The level of investment, I_t from the individuals who are alive at period t is used in the production process at period $t + 1$: $K_{t+1} = I_t$. We suppose that individuals who are in retirement do not work. Thus, only individuals at age 0 are workers and they offer inelastically one unit of labor. Moreover, there are not endowment at ages 1 and 2, $W_1 = W_2 = 0$. Individuals born at t receive, when they are young, the competitive wage, W_t, such that:

$$W_t N_{0t} = F_2'(K_t, N_{0t}) = F(K_t, N_{0t}) - K_t F_1'(K_t, N_{0t}).$$

The risk-free interest rate is remunerate to the marginal productivity of capital:

$$R_t = F_1'(K_t, N_{0t})$$

The capital market equilibrium is given by:

$$K_{t+1} = N_{0t} W_t + p_1 N_{0t-1}[A_{1t} + B_{t-1}]$$

By using the production function in its intensive form, $f(k_t) = F(k_t, 1)$ with $k_t = K_t/N_{0t}$, we simply obtain:

$$W_t = f(k_t) - k_t f'(k_t) \text{ and } R_t = f'(k_t).$$

The capital market equilibrium becomes:

$$(1+n) k_{t+1} = f(k_t) - k_t f'(k_t) + \frac{p_1}{1+n}[A_{1t} + B_{t-1}], \tag{50}$$

where A_{1t} and B_{t-1} are given by the implicit equations of individuals behaviors. It is then simple to obtain an existence result similar to the one that lies in Proposition 2. Consequently, the results on efficiency still hold.

4 Conclusion

To be done.

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